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ТЕОРИЯ, ИССЛЕДОВАНИЕ, ПРОЕКТИРОВАНИЕ И ВЫБОР ОПТИМАЛЬНЫХ ПАРАМЕТРОВ ИНТЕЛЕКТУАЛЬНЫХ ИНЖЕНЕРНЫХ СИСТЕМ. МЕТОДЫ МНОГОМЕРНОЙ МАТЕМАТИКИ

THEORY, RESEARCH, DESIGN AND SELECTION OF OPTIMAL PARAMETERS OF INTELLIGENT ENGINEERING SYSTEMS. METHODS OF MULTIDIMENSIONAL MATHEMATICS

Аннотация. Цель работы представление теории и методов Многомерной математики. Построение математической модели инженерной системы (на примере структуры материала) в виде векторной задачи математического программирования на стадии проектирования. Выбор оптимальных параметров в многопараметрической и многофункциональной инженерной системы на базе многомерной математики. Геометрическая интерпретация результатов решения N-мерной инженерной системы в двухмерной системе в относительных и физических единицах. В рамках теории векторной оптимизации представлены принципы оптимальности решения векторных задач при равнозначных критериях и при заданном приоритете критерия https://rdcu.be/bhZ8i.

(Работа "Vector optimization with equivalent and priority criteria" Springer Nature распространяется бесплатно.).

В работе на базе векторной оптимизации разработана методология проектирования инженерных систем и технологий путем: 1) построения *математической* модели инженерной системы в условиях определенности и неопределенности; разработки конструктивных методов решения векторной задачи; 2) представлено построение численной модели выбора оптимальных параметров сложной инженерной системы (материала сложной структуры: много параметрической и много функциональной); 3) представлена численная реализация модели структуры материала при равнозначных критериях; 4) представлена численная реализация модели структуры материала при заданном приоритете любого критерия; 5) представлена геометрическая интерпретация результатов решения при проектировании в трехмерной системе координат четырех характеристик (критериев) в относительных и физических единицах.

Abstract. The purpose of the work is to present the theory and methods of multidimensional mathematics in the process of research, design and selection of optimal parameters of engineering systems. The theory and methods of multidimensional mathematics can be the mathematical apparatus of computational intelligence within the framework of artificial intelligence. In mathematical models of engineering systems, purposefulness is formed in the form of a set of characteristics (criteria), i.e. models are represented by vector optimization problems. Within the framework of the theory of vector optimization, the principles of optimality of solving vector problems with equivalent criteria and with a given priority of the criterion are presented https://rdcu.be/bhZ8i (The work "Vector optimization with equivalent and priority criteria» by Springer Nature is distributed free of charge.).

In the work on the basis of vector optimization, the methodology for designing multiparameter and multifunctional engineering systems is presented for the first time by: 1) building a mathematical model of an engineering system under conditions of certainty and uncertainty; development of constructive methods for solving a vector problem; 2) the construction of a numerical model for the selection of optimal parameters of a complex engineering system (material of a complex structure: many parametric and many functional) is presented; 3) the numerical implementation of the model of the structure of the material with equivalent criteria is presented; 4) a numerical implementation of the material structure model is presented at a given priority of any criterion; 5) presents a geometric interpretation of the results of the solution when designing four characteristics (criteria) in relative and physical units in a three-dimensional coordinate system.

Ключевые слова: Многомерная математика, Теория векторной оптимизации, Методы принятия оптимальных решений, Моделирование инженерной системы, Геометрическая интерпретация многомерных систем.

Keywords: Multidimensional Mathematics, Vector Optimization, Methods for Solving Vector Problems; Theory of decision-making; Modelling of the production system; Modelling of the technical system.

Introduction

When studying the development of engineering systems and technologies, it turns out that they (systems) depend on a certain numerical set of functional characteristics, which together determine the multidimensionality of the system under study. This multidimensionality must be taken into account at the design and modelling stage. The analysis and study of many engineering and technologies systems has shown that an improvement in one of the characteristics of the system leads to a deterioration in other characteristics. And to improve the functioning of the system:

First, it is necessary to solve the problem in which in the system under study one subset of characteristics (criteria) was aimed at increasing the numerical value (maximization), and the second subset of the characteristics (criteria) of the system was aimed at decreasing the numerical value (minimization);

Secondly, it is necessary that all the characteristics are improved together.

At present, there is a solution to single-criteria optimization, which can be interpreted as univariate optimization.

The study of multi-criteria problems began more than a hundred years ago in the work of Pareto V. [1]. Over the past three decades, a large number of monographs and individual articles have been devoted to methods for solving vector (multi-criteria) problems. This is due to the widespread use of these methods in solving practical problems. The analysis of methods and algorithms for solving multi-criteria problems in accordance with its classification is presented in a number of works [6, 10, 22, 39, 45]: methods for solving multi-criteria problems based on the collapse of criteria with weight coefficients [3, 5, 25, 27]. The study of multi-criteria optimization was carried out both at the theoretical level by foreign [3, 25-30] and Russian authors [4-23], and at the solution of practical problems, first in the field of economics [31-45], and then in the field of engineering systems [6-21, 46-49].

The set of criteria in a multi-criteria optimization problem can be represented as a vector with which mathematical operations can be performed, hence the vector optimization problems or vector problems of mathematical programming.

Solving the problem of vector optimization is due to a number of difficulties, moreover, of a conceptual nature, and the main one is to understand: "what does it mean to solve the problem of vector optimization". To solve a problem with multiple criteria, it is mathematically necessary to create, first, axiomatics and principles of optimality, showing any user in what way one solution is better than another solution, and what is the optimal solution of a multi-criteria (i.e., with many characteristics) optimization problem. And at the next stage, on the basis of axiomatics and the principles of optimality, the development of constructive methods for solving multi-criteria problems.

***The purpose of the work* is** to present the theory and methods of multidimensional mathematics in the process of research, design and selection of optimal parameters of engineering systems and technologies, in which the purposefulness of the mathematical model of an engineering system is formed in the form of a set of characteristics (criteria).

In the work, within the framework of the theory of vector optimization, axioms are formulated and the principles of optimality for solving vector problems with equivalent criteria and with a given priority of the criterion are presented, as well as constructive methods for solving vector optimization problems are presented. The applied part presents constructive methods for making optimal decisions, methods for solving vector problems for modeling engineering systems, which are described by a set of functional characteristics.

In terms of organization, the first chapters present the vector problem of mathematical programming, [6-22], for the solution of which axiomatics is formulated and the principles of optimality are presented. Within the framework of the theory of vector optimization, methods for solving vector problems with equivalent criteria and with a given priority of the criterion have been developed. The presented constructive methods for solving vector problems of mathematical programming allow us to make a decision, firstly, with equivalent criteria, and secondly, with a given priority of the criterion of one or another criterion.

In the field of engineering systems, which include technical systems [11-14, 18, 19], technological processes [15, 21], materials [17].

In the work, on the basis of vector optimization, a methodology for designing engineering systems has been developed by: 1) building a mathematical model of an engineering system under conditions of certainty and uncertainty; development of constructive methods for solving a vector problem; 2) the construction of a numerical model for the selection of optimal parameters of a complex engineering system (material of a complex structure: many parametric and many functional) is presented; 3) the numerical implementation of the model of the structure of the material with equivalent criteria is presented; 4) a numerical implementation of the material structure model is presented at a given priority of any criterion; 5) presents a geometric interpretation of the results of the solution when designing four characteristics (criteria) in relative and physical units in a three-dimensional coordinate system.

1. Mathematical Modeling of Engineering Systems Based on Vector Optimization Problems

As an object of research, we consider Engineering and technologies systems, which include "technical systems", "technological processes", "materials", [18 and 19]. The engineering system study is carried out,

first, under conditions of certainty, when data on functional characteristics of the engineering system are known;

second, in uncertainty conditions where discrete values of individual characteristics are known; there are also known data on limitations imposed on the operation of the system.

1.1. Mathematical Model of a Technical System for Making an Optimal Management Decision

1.1.1. Problems of Modeling in the Design of New Technical Systems

When researching and designing new technical systems, the problem of building a mathematical model, evaluating the results of modeling and making an optimal decision based on them arises. Similar problems arise on already created technical systems (TS), which during operation must be constantly upgraded and replaced by more advanced models. The problem of modeling the vehicle (including using multi-criteria (vector) optimization) is given great attention, firstly, in domestic science, starting with the leading scientific schools of the Academy of Sciences of the USSR to the present time [4-23, 38-45], which have made a great contribution to the application of multi-criteria optimization methods, and, secondly, in foreign scientific activity, both theoretical [1, 2, 3, 24-33] and applied aspects [34-38].

When constructing a mathematical model TS, there may be situations when the functional dependence of each characteristic and constraint on the parameters of the TS is known, such a model is usually called a model in conditions of complete certainty (well-structured tasks) - while using the methods presented in the fourth chapter. Situations where there is insufficient information about the functional dependence of each characteristic and constraint on the parameters are defined as modeling under conditions of complete or partial uncertainty (poorly structured tasks) [12 - 18]. In this case, the elimination of uncertainty can go in two directions: the first is related to the use of subjective assessments and the preference of the decision-maker (LPR) in assessing the options for possible solutions [22]; the second direction, determined by the qualitative and quantitative descriptions of the TS, is characterized by the use of mathematical methods for transforming information - initial data that can be obtained, for example, on the principle of "input-output". Within the framework of the second direction, this study was carried out. Currently, there are a large number of methods for evaluating experimental data and making decisions based on [34], but all of them have certain drawbacks and, above all, those related to the incommensurability of the criteria, the analysis is presented in the works [6, 10, 22], as a result, the decision is made on an intuitive basis. Therefore, it is important to develop new methods for modeling the vehicle, evaluating the initial data and making decisions based on them.

The purpose of this section is to develop a methodology for building a mathematical model of a technical system in the form of a vector problem of mathematical programming.

1.1.2. Mathematical model of technical systems in the conditions of certainty

The technical system which functioning depends on $\boldsymbol{N}$ - a set of design data is considered:

$X = \{x_1\ x_2 \ldots x_N\}$, or $X = \{x_j,\ j = \overline{1,N}\}$, N - number of parameters, each of which lies in the set limits: $x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1,N}$, or $X^{min} \leq X \leq X^{max}$, where x_j^{min}, $x_j^{max}, \forall\, j \in N$ - lower and top limits of change of the vector of parameters of the technical system.

The result of functioning of the technical system is defined by a set $\boldsymbol{K}$ to technical characteristics of $f_k(X), k = \overline{1,K}$, which functionally depend on design data $X = \{x_j, j = \overline{1,N}\}$, in total they represent a vector function:

$F(X) = (f_1(X)\ f_2(X) \ldots f_K(X))^T$, on which functional limitations are imposed:

$f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1,K}$ or $G(X) \leq B$.

The set of characteristics (criteria) to is subdivided into two subsets $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$: $\boldsymbol{K} = \boldsymbol{K}_1 \cup \boldsymbol{K}_2$ is a subset of technical characteristics, the numerical values of which are desirable to obtain as high as possible: $f_k(X) \to max,\ k = \overline{1,K_1}$.

$\boldsymbol{K}_2$ - it subsets of technical characteristics which numerical sizes it is desirable to receive as it is possible below: $f_k(X) \to min,\ k = \overline{1,K_2}, \boldsymbol{K}_2 \equiv \overline{K_1 + 1, K}$.

The mathematical model should, firstly, the purposes of the technical system which are presented by the characteristics of $F(X) = \{f_k(X), k = \overline{1,K}\}^T$ secondly, to consider $X^{min} \leq X \leq X^{max}$ restrictions. The mathematical model of the technical system which solves in general a problem of a choice of the optimum design decision (a choice of optimum parameters), we will present in the form of a vector problem of mathematical programming.

$$Opt\ F(X) = \{max\, F_1(X) = \{max\, f_k(X), k = \overline{1,K_1}\}, \tag{1a}$$

$$min\, F_2(X) = \{min\, f_k(X), k = \overline{1,K_2}\}\}, \tag{2a}$$

$$\text{the restrictions: } G(X) \leq 0, f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1,K}, \tag{3a}$$

$$x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1,N}, \tag{4a}$$

where X is the vector of controlled variables (constructive parameters);

$F(X) = \{f_k,\ k = \overline{1,K}\}$ – vector criterion which everyone a component submits the characteristic of the TS (1a)-(2a) which is functionally depending on the vector of variables X;

in (3a) $G(X) = \{g_1(X)\ g_2(X) \ldots\ g_M(X)\}^T$ – vector-a function of the restrictions imposed on functioning of the technical system, $\boldsymbol{M}$ – a set of restrictions.

Restrictions are defined proceeding in them technological, physical and to that similar processes and can be presented by functional restrictions, for example:

$f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1,K}$.

It is supposed that the $f_k,\ k = \overline{1,K}$ functions are differentiated and convex, $g_i(X),\ i = \overline{1,M}$, are continuous, and (3a)-(4a) set of admissible points of $\boldsymbol{S}$ set by restrictions isn't empty and represents a compact: $\boldsymbol{S} = \{X \in \boldsymbol{R}^n | G(X) \leq 0, X^{min} \leq X \leq X^{max}\} \neq \emptyset$.

Criteria and restrictions (1a)-(4a) form mathematical model of technical system. It is required to find such vector of the $X^o \in \boldsymbol{S}$ parameters at which everyone a component the vector - functions $F(X)$: $F_1(X) = \{f_k X), k = \overline{1,K_1}\}$ accepts the greatest possible value, and a vector - functions $F_2(X) = \{f_k X), k = \overline{1,K_2}\}$ are accepted by the minimum value.

1.1.3. Problems of Constructing a Model of a Technical System in the Conditions of Certainty and Uncertainty in the Aggregate

When constructing a mathematical model of TS (1a)-(4a), two options are possible. The first option, when the functional dependence of each criterion (1a)-(2a), restrictions (3a)-(4a) on the parameters imposed on functioning is known, it is customary to call it a model under conditions of certainty, which are represented by model (1a)-(4a). The second option is when there is insufficient information about the functional dependence of each criterion (1a)-(2a) and restrictions (3a)-(4a) on the parameters. Only discrete (usually experimental) data are known:

$X_i = \begin{vmatrix} x_{i1}, \ldots, x_{iN} \\ \ldots \\ x_{i1}, \ldots, x_{iN} \end{vmatrix}$, $i = \overline{1,M}$, where N is a set of parameters, M is a set of data, or $X_i = \{x_{ij}, j = \overline{1,N}, i = \overline{1,M}\}$. A corresponding discrete set of characteristics is also presented:

$$I_i(X) = \{(y_k(X_i,\ i = \overline{1,M})\}^T = \begin{vmatrix} y_{i1}(X_i), \ldots, y_{iK}(X_i) \\ \ldots \\ y_{i1}(X_i), \ldots, y_{iK}(X_i) \end{vmatrix}.$$

Such a state is represented by the modelling of a technical system under conditions of uncertainty. To obtain data $I_i(X) = \{(y_k(X_i,\ i = \overline{1,M})\}^T$ experimental studies of the technical system are carried out on the principle of "input-output", while the problem of decision-making under conditions of uncertainty is formed and the problem of choosing the optimal estimate based on the data obtained arises..

1.1.4. Constructing a Mathematical Model of a Technical System in Conditions of Certainty and Uncertainty in the Aggregate

In real life, the conditions of certainty and uncertainty are combined. The model of the technical system should also reflect these conditions. Using the symbols of the mathematical model, we transform the model (1a) - (4a) taking into account the uncertainty conditions, as a result we get:

"Model of a technical system in conditions of certainty and uncertainty":

$$Opt\ F(X) = \{max\ F_1(X) = \{max\ f_k\ (X), k = \overline{1, K_1^{def}}\}, \quad (5a)$$

$$max\ I_1(X) \equiv \{max\ y_k\ (X_i,\ i = \overline{1,M})\}^T, k = \overline{1, K_1^{unc}}\}, \quad (6a)$$

$$min\ F_2(X) = \{min\ f_k\ (X), k = \overline{1, K_2^{def}}, \quad (7a)$$

$$min\ I_2(X) \equiv \{min\ y_k\ (X_i,\ i = \overline{1,M})\}^T, k = \overline{1, K_1^{unc}}\}\}, \quad (8a)$$

$$\text{the restrictions } f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1,K}, \quad (9a)$$

$$x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1, N}, \quad (10a)$$

where $X = \{x_j, \ j = \overline{1, N}\}$ - vector of controlled variables (parameters) TS;

$F(X) = \{F_1(X) \ F_2(X) \ I_1(X) \ I_2(X)\}$- a vector criterion, each component of which represents a vector of criteria (characteristics) of a technical system (5a) - (8a), which functionally depend on the discrete values of the variable vector , where in X (5a) and (7a) K_1^{def}, K_2^{def} (*definiteness*), а в (6a) и (8a) K_1^{unc}, K_2^{unc} (*uncertainty*) a set of criteria max and min formed in conditions of certainty and certainty.

1.1.5. Example. Simulation and Optimization of Parameters of Magnetoelectric Linear Inductor Electric Motors (LD) on DC

The content class of technical systems that can be represented by a vector problem (5a) - (10a) can be attributed to a fairly large number of their tasks from various sectors of the state economy: electrical, aerospace, metallurgical (selection of the optimal structure of the material), chemical, etc.

As an example, let's present the work of V. L. Levitsky "Modeling and optimization of parameters of magnetoelectric linear inductor electric motors (LD) of direct current" [7, pp.50-120]. A construct was designed - forced LD (FLD), the model of which was reduced to a vector problem of mathematical programming (1a) - (4a). In this case, the vector of design parameters $X = \{x_1, x_2, \dots, x_5\}$ consisted of: x_1 – air gap δ, x_2 – toothed pitch, x_3 – numbers of teeth, x_4 – hub height, x_5 – pole overlap coefficient. Vector of design criteria $F(X) = f(X), \ p(X), \vartheta(X)$, ... included: $f(X)$ – rated tractive effort, $p(X)$ – rated power, $\vartheta(X)$ – nominal efficiency, etc. p., a total of ten indicators. To construct the dependencies $F(X)$ on these design parameters, the central orthogonal plan of the second order was used [7 p.96]. From another branch, we will present the work [23] - "... multi-criteria optimization of static modes of mass transfer processes on the example of absorption in the production of gas separation". Thus, experimental data, both from the FLD problem and similar vehicles from other industries, can be presented in the form of a theoretical (systemic) problem (1a) - (4a).

In this article, the technical system is considered in statics. But technical systems can be considered in dynamics, using differential-difference methods of transformation, conducting research for a small discrete period Δt ∈ T.

1.2. Mathematical modeling of the technological process for making an optimal management decision

1.2.1. Mathematical model of a technological process in conditions of certainty

Please As an object of a research we use "technological process". The problem of decision-making in technology in the production of products is formulated in accordance with the works [15].

We consider a technological process (e.g., Hybrid laser arc welding, [15], in which ze41-T5 alloy was chosen as the material to be welded with AZ61 alloy as the filler material).

Activity of technological process depends on a particular set of conditions - design data:

$X = \{x_1, x_2, \dots, x_N\}^T$, or $X = \{x_j, \ j = \overline{1, N}\}$,

(for example: laser powers; speeds of movement; feed rates of a wire; current; frequencies) we Will designate. Let's denote ***N***-set of constructive parameters. Each parameter of the technological process lies in the given limits:

$x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1, N}$, or $X^{min} \leq X \leq X^{max}$,

where x_j^{min}, $x_j^{max}, \forall j \in N$ - the lower and top limits of change of a vector of parameters of technological process, N is the number of parameters.

The result of functioning is defined by a set of running characteristics $f_k(X), k = \overline{1, K}$, which functionally depend on design data of technological process of $X = \{x_j, j = \overline{1, N}\}$, (for example: the depth of the weld; underfill; percentage defect; total accumulated pore length). In total all running characteristics represent a vector function:

$F(X) = (f_1(X) \ f_2(X) \dots \ f_K(X))^T$, or $F(X) = \{f_k(X), k = \overline{1, K}\}^T$, where $\boldsymbol{K}$ – a set (number) of technological characteristics (criteria).

The set of characteristics $\boldsymbol{K}$ is subdivided into two subsets of $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$: $\boldsymbol{K} = \boldsymbol{K}_1 \cup \boldsymbol{K}_2, \boldsymbol{K}_1 \subset \boldsymbol{K}, \boldsymbol{K}_2 \subset \boldsymbol{K}$.

$\boldsymbol{K}_1$ is a subset of technological characteristics, the numerical values of which are desirable to obtain as high as possible: $f_k \to max,\ k = \overline{1, K_1}$.

$\boldsymbol{K}_2$ are subsets of technological characteristics, the numerical values of which are desirable to obtain as low as possible: $f_k \to min,\ k = \overline{1, K_1+1}, \boldsymbol{K}_2 \equiv \overline{K_1+1, K}$.

Mathematical model has to reflect, first, the purposes of the technological process which are presented by the characteristics of $F(X)$, and secondly, to take into account the constraints of $X^{min} \le X \le X^{max}$. The mathematical model of the technological process solving in general a problem of the choice of optimum parameters of the technological process can be presented in the form of a vector problem of mathematical programming.

$$Opt\ F(X) = \{max\ F_1(X) = \{max\ f_k(X), k = \overline{1, K_1}\}, \quad (1b)$$

$$min\ F_2(X) = \{min\ f_k(X), k = \overline{1, K_2}\}\}, \quad (2b)$$

$$\text{the restrictions: } G(X) \le 0, f_k^{min} \le f_k(X) \le f_k^{max}, k = \overline{1, K}, \quad (3b)$$

$$x_j^{min} \le x_j \le x_j^{max}, j = \overline{1, N}, \quad (4b)$$

where X is the vector of operated variable (design parameters) of the technological process;

$F(X) = \{f_k(X),\ k = \overline{1, K}\}$ - vector criterion which each component represents the characteristic of the technical system (1*b*)-(2*b*) which is functionally depending on a vector of variables X;

in (3.3*b*) $G(X) = \{g_1(X)\ g_2(X) \dots\ g_M(X)\}^T$ is a vector function of the restrictions imposed on functioning of the technological process, $\boldsymbol{M}$ is a set of constraints. The constraints are defined flowing past in them technological, physical and to that similar processes and can be presented by the functional restrictions, for example: $f_k^{min} \le f_k(X) \le f_k^{max}, k = \overline{1, K}$.

It is supposed that the functions $f_k(X), k = \overline{1, K}$ are differentiated and convex, $g_i(X),\ i = \overline{1, M}$ are continuous, and $\boldsymbol{S} = \{X \in R^n | G(X) \le 0, X^{min} \le X \le X^{max}\} \ne \emptyset$.

The (1*b*)-(4*b*) ratios form mathematical model of the technological process. It is required to find such vector of the $X^o \in \boldsymbol{S}$ parameters at which each component of a vector function of $F_1(X) = \{f_k X), k = \overline{1, K_1}$ accepts the greatest possible value, and vector functions of $F_2(X) = \{f_k X), k = \overline{1, K_2}$ accepts minimum value.

1.2.2. Mathematical model of the technological process under conditions of certainty and uncertainty in the aggregate

The mathematical model of the technological process (1*b*) - (4*b*) is made under conditions of certainty. In real life, the conditions of certainty and uncertainty are combined. The model of the technological process should also reflect these conditions. Using the designations of the mathematical model, we transform the model (1*b*) - (4*b*) taking into account the conditions of uncertainty, as a result, we get:

"Model of the technological process in conditions of certainty and uncertainty":

$$Opt\ F(X) = \{max\ F_1(X) = \{max\ f_k(X), k = \overline{1, K_1^{def}}\}, \quad (5b)$$

$$max\ I_1(X) \equiv \{max\ y_k(X_i,\ i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}, \quad (6b)$$

$$min\ F_2(X) = \{min\ f_k(X), k = \overline{1, K_2^{def}}, \quad (7b)$$

$$min\ I_2(X) \equiv \{min\ y_k(X_i,\ i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}\}, \quad (8b)$$

$$\text{the restrictions: } f_k^{min} \le f_k(X) \le f_k^{max}, k = \overline{1, K}, \quad (9b)$$

$$x_j^{min} \le x_j \le x_j^{max}, j = \overline{1, N}, \quad (10b)$$

where $X = \{x_j,\ j = \overline{1,N}\}$ - vector of controlled variables (parameters) of the technological process;

$F(X) = \{F_1(X)\ F_2(X)\ I_1(X)\ I_2(X)\}$ - a vector criterion, each component of which represents a vector of criteria (characteristics) of the technological process (5*b*) - (8*b*), which functionally depend on the discrete values of the variable vector , where X in (5*b*) and (3.7*b*) K_1^{def}, K_2^{def} (*definiteness*), а в (6*b*) и (8*b*) K_1^{unc}, K_2^{unc} (*uncertainty*) a set of criteria max and min formed in conditions of certainty and certainty.

1.3. Mathematical Model of Material Structure for Optimal Management Decision Making

1.3.1. The Problem of Material Structure Modeling

Chemical composition of material of a product is defined (on unit of volume, weights) by the percentage maintenance of some set of components of material which are equal in the sum to hundred percent. The composition of material, is characterized by a particular set of the functional characteristics which include mechanical and physical and chemical characteristics of materials. One group of properties (the functional characteristics) of material is characterized by the fact that it is desirable to receive them on the numerical value as much as possible (for example, durability), other group of properties is characterized by the fact that is desirable to receive them on the numerical value less as. Improvement on one of these characteristics leads another to deterioration. In general, it is required to pick up such composition of material that all properties of material were as it is possible better in total.

1.3.2. Mathematical Model of Material Structure under Conditions of Certainty

Discusses the composition of the material, any product, technical system that depends on a number of material component: $Y = \{y_1, y_2, \dots, y_V\}$, where $\boldsymbol{V}$ is the set of components of the material, $Y = \{y_j, j = \overline{1,V}\}$, V is the number of components of which it can be made (fabricated) material, y_v is the size as a percentage *v*th of a material component, each of which lies in the given limits:

$y_v^{min} \leq y_v \leq y_v^{max},\ v = \overline{1,V}$, where y_v^{min}, y_v^{max}, $\forall v \in V$ are the lower and upper limits of the change in the vector of the material components.

$\sum_{v=1}^{V} y_v(t) = 100\%$, the sum of all the components of the material is one hundred percent.

The composition of material is estimated by set K physical properties of material:

$H(Y) = \{h_k(Y), k = \overline{1,K}\}$, which functionally depend on design data of $Y = \{y_j, j = \overline{1,V}\}^T$;

k is the index of a type of physical properties of material, $k = \overline{1,K}$, where K - number of types of properties (the functional characteristics) of material, we will present them in the form a vector – functions. $H(Y)$ *is* a vector function (vector criterion) having K a component function:

$H(Y) = \{h_k(Y), k = \overline{1,K}\}$.

The set physical properties of material $\boldsymbol{K}$ consists of sets of $\boldsymbol{K}_1$, a component of maximization and $\boldsymbol{K}_2$ of minimization; $\boldsymbol{K} = \boldsymbol{K}_1 \cup \boldsymbol{K}_2$;

$H_1(Y) = \{h_k(Y),\ k = \overline{1,K_1}\}$ is maximizing vector-criterion, K_1 – number of criteria, and $K_1 \equiv \overline{1,K_1}$ is a set of maximizing criteria. Let's further assume that $H_1(Y) = \{h_k(Y),\ k = \overline{1,K_1}\}$ is the continuous concave functions (we will sometimes call them the maximizing criteria);

$H_2(Y) = \{h_k(Y),\ k = \overline{1,K_2}\}$ is vector criterion in which each component is minimized, $K_2 \equiv \overline{K_1 + 1, K} \equiv \overline{1,K_2}$ - a set of minimization criteria, K_2 – number. We assume that $h_k(Y), k = \overline{1,K_2}$ is the continuous convex functions (we will sometimes call these the minimization criteria), i.e.: $\boldsymbol{K} = K_1 \cup K_2$, $K_1 \subset \boldsymbol{K}, \boldsymbol{K_2} \subset \boldsymbol{K}$.

We use characteristics of the material $H(Y) = \{h_k(Y), k = \overline{1,K}\}$ as criterion, and change limits imposed on each type of components as parametrical restrictions. We will present the mathematical model of material solving in general a problem of the choice of the optimal design solution (the choice of optimum structure of material) in the form of a vector problem of mathematical programming:

$$Opt\ F(X) = \{max\ H_1(X) = \{max\ h_k(X), k = \overline{1, K_1}\}, \quad (1c)$$

$$min\ H_2(X) = \{min\ h_k(X), k = \overline{1, K_2}\}\}, \quad (2c)$$

the restrictions: $G(Y) \leq 0,\ h_k^{min} \leq h_k(X) \leq h_k^{max}, k = \overline{1, K}$, (3c)

$$\sum_{v=1}^{V} y_v(t) = 100\%,\ y_v^{min} \leq y_v \leq y_v^{max}, v = \overline{1, V}, \quad (4c)$$

where $Y = \{y_j, j = \overline{1, V}\}$ is a vector of the operated variables (a material component);

$H(Y) = \{h_k(Y), k = \overline{1, K}\}$ *is* vector criterion which each function submits the characteristic (property) of material which is functionally depending on a vector of variables *Y*;

$G(Y) = \{g_1(Y), \dots, g_M(Y)\}^T$ is a vector function of the restrictions imposed on structure of material, $\boldsymbol{M}$ – a set of restrictions.

It is supposed that the functions $H(Y) = \{h_k(Y), k = \overline{1, K}\}$ are differentiated and convex, $G(Y) = \{g_i(Y), i = \overline{1, M}\}^T$ are continuous, and (1*c*)-(4*c*) set of admissible points of $\boldsymbol{S}$ set by restrictions are not empty and represents a compact:

$\boldsymbol{S} = \{X \in \boldsymbol{R}^n | G(X) \leq 0, X^{min} \leq X \leq X^{max}\} \neq \emptyset$.

The relations (1*c*)-(4*c*) form mathematical model of material. It is required to find such vector of the $Y^o \in \boldsymbol{S}$ parameters at which each component (characteristic) the vector - functions $H_1(Y)$ accepts the greatest possible value, and a vector - functions $H_2(Y)$ accepts minimum value:

$H_1(Y) = \{h_k(Y),\ k = \overline{1, K_1}\}, H_2(Y) = \{h_k(Y),\ k = \overline{1, K_2}\}$.

In set the mathematical model of material (1*c*)-(4*c*) can be treated as systems approach to material research.

1.3.3. Mathematical model of the structure of the material under conditions of certainty and uncertainty in the aggregate

The mathematical model of the structure of the material (1*c*)-(4*c*) is made under conditions of certainty. In real life, the conditions of certainty and uncertainty are combined. The model of the technological process should also reflect these conditions. Using the designations of the mathematical model, we transform the model (1*c*)-(4*c*) taking into account the conditions of uncertainty, as a result, we get:

"Model of the structure of the material in conditions of certainty and uncertainty":

$$Opt\ F(X) = \{max\ H_1(X) = \{max\ h_k(X), k = \overline{1, K_1^{def}}\}, \quad (5c)$$

$$max\ I_1(X) \equiv \{max\ y_k(X_i,\ i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}, \quad (6c)$$

$$min\ H_2(X) = \{min\ h_k(X), k = \overline{1, K_2^{def}}, \quad (7c)$$

$$min\ I_2(X) \equiv \{min\ y_k(X_i,\ i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}\}, \quad (8c)$$

the restrictions: $G(Y) \leq 0,\ h_k^{min} \leq h_k(X) \leq h_k^{max}, k = \overline{1, K}$, (9c)

$$\sum_{v=1}^{V} y_v(t) = 100\%,\ y_v^{min} \leq y_v \leq y_v^{max}, v = \overline{1, V}, \quad (10c)$$

where $X = \{x_j,\ j = \overline{1, N}\}$ - vector of controlled variables (parameters) of the technological process;

$F(X) = \{F_1(X)\ F_2(X)\ I_1(X)\ I_2(X)\}$- a vector criterion, each component of which represents a vector of criteria (characteristics) of a technical system (5*c*) - (8*c*), which functionally depend on the discrete values of the variable vector , where in *X* (5*c*) and (7*c*) K_1^{def}, K_2^{def} (*definiteness*), а в (6*c*) и (8*c*) K_1^{unc}, K_2^{unc} (*uncertainty*) a set of criteria max and min formed in conditions of certainty and certainty.

2. Multidimensional Mathematics. Theory, methods.

Mathematical models under conditions of certainty and uncertainty: technical system (1a)-(4a), (5a)-(10a), technological process (1b)-(4b), (5b)-(10b), material structure (1c)-(4c), (5c)-(10c) are represented by vector problems of mathematical programming (VPMP). Further development of the study of works on the theory of vector optimization led to the formation of "Multidimensional Mathematics". In this aspect, we will present four sections:

Vector problem of mathematical programming;

Axiomatics, the principle of optimality and constructive methods of vector optimization with equivalent criteria;

Axiomatics, the principle of optimality and constructive methods of vector optimization with a given priority of the criterion;

Applied Multidimensional Mathematics: Vector Problem of Nonlinear Programming as a Model for the Development of Engineering Systems.

2.1. A vector problem of mathematical programming

2.1.1. Introduction to Multidimensional Mathematics.

As a representative of multidimensional mathematics, we will formulate the problem of mathematical programming, represented by a set of functions that determine the multidimensionality of the object under study. Each function of this set of functions has a different target orientation: maximization or minimization, which in the aggregate change on a certain (not empty and closed) set of variables (parameters).

Without breaking the generality, a set of functions can be represented in the form of a vector of functions. As a result, we get a vector problem of mathematical programming. On the basis of the vector optimization problem, we present the theoretical problems necessary for its solution, which include axiomatics (theoretical foundations), the principle of optimality and constructive methods for solving vector problems with equivalent criteria and a given priority of the criterion, [6, 15, 29, 31, 33, 44].

2.1.2. A vector problem of mathematical programming

A vector problem in mathematical programming (VPMP) is a standard mathematical-programming problem including a set of criteria, which, in total, represent a vector of criteria.

It is important to distinguish between uniform and non-uniform VPMP:

A uniform maximizing VPMP is a vector problem in which each criterion is directed towards maximizing;

A uniform minimizing VPMP is a vector problem in which each criterion is directed towards minimizing;

A non-uniform VPMP is a vector problem in which the set of criteria is shared between two subsets (vectors) of criteria (maximization and minimization respectively), e.g., non-uniform VPMP are associated with two types of uniform problems.

According to these definitions, we will present a vector problem in mathematical programming with non-uniform criteria [6, 20, 22] in the following form:

$$Opt\ F(X) = \{\max F_1(X) = \{\max f_k\,(X),\ k = \overline{1, K_1}\}, \quad (1)$$

$$\min F_2(X) = \{\min f_k\,(X),\ k = \overline{1, K_2}\}\}, \quad (2)$$

$$f_k^{min} \le f_k(X) \le f_k^{max},\ k = \overline{1, K},\ G(X) \le B, \quad (3)$$

$$X^{min} \le X \le X^{max}, \quad (4)$$

where $X = \{x_j,\ j = \overline{1, N}\}$ is a vector of material variables, *N*-dimensional Euclidean space of R^N, $X^{min} \le X \le X^{max}$ - Parametric Constraints;

$F(X)$ *is* a vector function (vector criterion) having *K* – a component functions, (*K* - set power ***K***), $F(X) = \{f_k(X),\ k = \overline{1, K}\}$. The set ***K*** consists of sets of $\boldsymbol{K}_1$, a component of maximization and $\boldsymbol{K}_2$ of minimization; $\boldsymbol{K} = \boldsymbol{K}_1 \cup \boldsymbol{K}_2$ therefore we enter the designation of the operation "*opt*," which includes *max* and *min*;

$F_1(X) = \{f_k(X),\ k = \overline{1, K_1}\}$ is maximizing vector-criterion, K_1 – number of criteria, and $\boldsymbol{K}_1 \equiv \overline{1, K_1}$ is a set of maximizing criteria (a problem (1), (3), (4) represents VPMP with the homogeneous maximizing criteria). Let's further assume that $f_k(X), k = \overline{1, K_1}$ is the continuous concave functions (we will sometimes call them the maximizing criteria);

$F_2(X) = \{f_k(X),\ k = \overline{1, K_2}\}$ is vector criterion in which each component is minimized, $\boldsymbol{K}_2 \equiv \overline{K_1 + 1, K} \equiv \overline{1, K_2}$ - a set of minimization criteria, $\boldsymbol{K}_2$ – number, (the problems (2)-(4) are VPMP with the homogeneous minimization criteria). We assume that $f_k(X), k = \overline{1, K_2}$ is the continuous convex functions:

$$\boldsymbol{K}_1 \cup \boldsymbol{K}_2 = \boldsymbol{K},\ \boldsymbol{K}_1 \subset \boldsymbol{K},\ \boldsymbol{K}_2 \subset \boldsymbol{K}. \quad (5)$$

f_k^{min}, f_k^{max} – Minimum and maximum value of the *k*th criterion in restrictions,

$G(X) \leq B, X \geq 0$ is standard restrictions, $g_i(X) \leq b_i, i = 1, \dots, M$ where b_i - a set of real numbers, and $g_i(X)$ are assumed continuous and convex.

$$\boldsymbol{S} = \{X \in \boldsymbol{R}^n | X \geq 0, G(X) \leq B, X^{min} \leq X \leq X^{max}\} \neq \emptyset, \quad (6)$$

where the set of admissible points set by restrictions (3)-(4) is not empty and represents a compact.

The vector minimization function (criterion) $F_2(X)$can be transformed to the vector maximization function (criterion) by the multiplication of each component of $F_2(X)$ to minus unit. The vector criterion of $F_2(X)$ is injected into VPMP (1)-(4) to show that, in a problem, there are two subsets of criteria of $\boldsymbol{K}_1$, $\boldsymbol{K}_2$ with, in essence, various directions of optimization.

We assume that the optimum points received by each criterion do not coincide for at least two criteria. If all points of an optimum coincide among themselves for all criteria, then we consider the decision trivially.

2.1.3. Axioms and Axiomatic Methods

An axiom is a statement that does not require logical proof. On the basis of these statements (initial assumptions), one or another theory is built.

The axiomatic method is a method of constructing a scientific theory, in which the theory is based on some initial assumptions called the axioms of the theory. As a result, all other provisions of the theory are obtained as logical consequences of axioms [1, 2].

In mathematics, the axiomatic method originated in the works of ancient Greek geometers. An example of the axiomatic method is the ancient Greek scientist Euclid, whose axioms were laid down in his famous work "Elements".

The axiomatic method was further developed in the works of D. Hilbert in the form of the so-called method of system formalism. The general scheme of building an arbitrary formal system ("S") includes:

1. The language of the system ("S"), including the alphabet – this is a list of elementary symbols; the rules of formation (syntax) on which the formulas "S" are built.
2. Axioms of the "S" system, which represent a certain set of formulas.
3. Rules for the withdrawal of the "S" system [2].

In the application to the solution of the problem of vector optimization (multidimensional mathematics), axiomatics is divided into two sections: 1. Axiomatics of solving the vector optimization problem with equivalent criteria; 2. Axiomatics of solving the vector optimization problem with a given priority of criteria. Only with the construction of the initial axiomatics is it possible in the future to construct the principle of optimality and the resulting algorithms for solving vector problems of mathematical programming.

2.2. Axiomatics, the principle of optimality and constructive methods of vector optimization with the equivalent criteria

2.2.1. Axiomatics of Solving a Vector Optimization Problem with Equivalent Criteria

In accordance with the above interpretation, the language of the system of multidimensional mathematics includes: first, the normalization of criteria, second, the relative evaluation of criteria (functions), and, third, the minimum relative level.

Definition 1. (***Normalizing of the criterion***).

Normalizing criteria (mathematical operation: the shift plus rationing) presents a unique display of the function $f_k(X)\ \forall k \in K$, in a one-dimensional space of $\boldsymbol{R}^1$(the function $f_k(X)\ \forall k \in K$

represents a function of transformation from a N-dimensional Euclidean space of $\boldsymbol{R^N}$ in $\boldsymbol{R^1}$). To normalize criteria in vector problems, linear transformations will be used:

$f_k(X) = a_k f'_k(X) + c_k \forall k \in K$, or

$$f_k(X) = (f'_k(X) + c_k)/a_k \ \ \forall k \in K, \qquad (7)$$

where $f'_k(X)$, $k = \overline{1, K}$ - aged (before normalization) value of criterion; $f_k(X), k = \overline{1, K}$ - the normalized value, a_k, c_k - constants.

Normalization of criteria $f_k(X) = (f'_k(X) + c_k)/a_k \ \forall k \in$К is a simple (linear) invariant transformation of a polynomial, as a result of which the structure of the polynomial remains unchanged. In the optimization problem, the normalization of criteria $f_k(X) = (f'_k(X) + c_k)/a_k \ \forall k \in$К does not affect the result of the solution. Indeed, if the convex optimization problem is solved:

$$\boldsymbol{max}_{X \in S}\ f(X), \text{ then at the optimum point } X^* \in S\text{: } \frac{df(X^*)}{dX} = 0. \qquad (8)$$

In the general case (including the normalization of the criterion (1)), the problem is solved:

$$\boldsymbol{max}_{X \in S}\ (a_k f'_k(X) + c_k), \qquad (9)$$

then at the optimum point $X^* \in S$:

$$\frac{d(a_k f(X^*) + c_k)}{dX} = a_k \frac{d(f(X^*))}{dX} + \frac{d(c_k)}{dX} = 0. \qquad (10)$$

The result is identical, i.e. the optimum point $X^*_k, k = \overline{1, K}$ is the same for non-normalized and normalized problems.

Definition 2. ***Relative evaluation of the function (criterion)***.

In the vector problem (1)-(4), normalize (7) of the form:

$$\lambda_k(X) = \frac{f_k(X) - f^0_k}{f^*_k - f^0_k}, \forall k \in \boldsymbol{K} \qquad (11)$$

is the relative estimate of a point $X \in \boldsymbol{S}$ kth criterion $f_k(X)$ - kth criterion at the point $X \in \boldsymbol{S}$;

f^*_k – value of the kth criterion at the point of optimum X^*_k, obtained in vector problem (1) - (4) of individual kth criterion; f^0_k is the worst value of the kth criterion (ant optimum) at the point X^0_k (Superscript 0 - zero) on the admissible set $\boldsymbol{S}$ in vector problem (1)-(4);

the task at max (1), (3), (4) the value of f^0_k is the lowest value of the kth criterion

$f^0_k = min_{X \in S} f_k(X) \ \forall k \in K_1$,

and task min (2), (3), (4) the value of f^0_k is the greatest value of the kth criterion:

$f^0_k = max_{X \in S} f_k(X) \ \forall k \in K_2$.

The relative estimate of the $\lambda_k(X) \ \forall k \in K$ is first, measured in relative units; secondly, the relative assessment of the $\lambda_k(X) \ \forall k \in K$: on the admissible set is changed from zero in a point of X^0_k: $\forall k \in \boldsymbol{K} \lim_{X \to X^0_k} \lambda_k(X) = 0$, to the unit at the point of an optimum of X^*_k:

$$\forall k \in \boldsymbol{K} \lim_{X \to X^*_k} \lambda_k(X) = 1\text{: } \forall k \in \boldsymbol{K}\ 0 \le \lambda_k(X) \le 1, X \in \boldsymbol{S}. \qquad (12)$$

As a result of this normalization, all the criteria of the VPMP are (1)-(4) are comparable in relative units, which allows comparing them with each other, using criteria for joint optimization.

Definition 3. ***The operation of comparing relative estimates of a function (criterion) with each other***.

Since any function (criterion) is represented in the relative estimates of the functions $\lambda_k(X) \ \forall k \in K$, which lie within the range of (12), it is possible to compare the relative estimates by numerical value. For comparison, the "subtraction" operation is used. If two functions (criteria) measured in the relative estimates $\lambda_{k=1}(X)$ and $\lambda_{k=2}(X) \ \forall k \in K$ are compared, then three situations are possible:

the first, when $\lambda_{k=1}(X) > \lambda_{k=2}(X)$;

the second, when $\lambda_{k=1}(X) = \lambda_{k=2}(X)$;

the third, when $\lambda_{k=1}(X) < \lambda_{k=2}(X)$.

The first and third situations are explored in Section 4.

This section examines the second situation.

2.2.2.

Axiomatics of Vector Optimization with Equivalent Criteria

Axiom 1. (*On the equivalence of criteria at an admissible point of a vector problem of mathematical programming*). In of vector problems of mathematical programming two criteria with the indexes $k \in \boldsymbol{K}, q \in \boldsymbol{K}$ shall be considered equivalent in $X \in \boldsymbol{S}$ point if relative estimates on kth and qth to criterion are equal among themselves in this point, i.e.

$\lambda_k(X) = \lambda_q(X), k, q \in \boldsymbol{K}.$

Explanation. If at point $X \in \boldsymbol{S}$ the functions (criteria) are equal to:

$\lambda_l(X) = 0{,}45\ l \in \boldsymbol{K}$ and $\lambda_q(X) = 0{,}45, q \in \boldsymbol{K}$ (i.e., 45% of its optimal value, which in relative units is equal to 1), then such criteria are not "equal" to each other, but are equivalent in their numerical value. And each of them carries its own functional meaning, which can be obtained using the normalization of criteria (11).

*Definition 4***.** (***Definition of a minimum level among all relative estimates of criteria***).

The relative level λ in a vector problem represents the lower assessment of a point of X∈S among all relative estimates of $\lambda_k(X), k = \overline{1,K}$:

$$\forall X \in \boldsymbol{S}\ \lambda \le \lambda_k(X), k = \overline{1,K}\ , \qquad (13)$$

the lower level for performance of a condition (13) in an admissible point of $X \in \boldsymbol{S}$ is defined by a formula

$$\forall X \in \boldsymbol{S}\ \lambda = \boldsymbol{min}_{k \in K} \lambda_k(X). \qquad (14)$$

Ratios (13) and (14) are interconnected. They serve as transition from operation (14) of definition of min to restrictions (13) and vice versa.

The level λ allows to unite all criteria in a vector problem one numerical characteristic of λ and to make over her certain operations, thereby, carrying out these operations over all criteria measured in relative units. The level λ functionally depends on the X ∈ $\boldsymbol{S}$ variable, changing X, we can change the lower level - λ. From here we will formulate the rule of search of the optimum decision.

2.2.3. The principle of optimality for vector optimization with the equivalent criteria

Definition 5. (***The principle of an optimality 1 with equivalent criteria***).

The vector problem of mathematical programming at equivalent criteria is solved, if the point of $X^o \in \boldsymbol{S}$ and a maximum level of λ^o (the top index o - optimum) among all relative estimates such that is found

$$\lambda^o = \boldsymbol{max}_{X \in S} \boldsymbol{min}_{k \in K} \lambda_k(X). \qquad (15)$$

Using interrelation of expressions (13) and (14), we will transform a maximine problem (15) to an extreme problem

$$\lambda^o = \boldsymbol{max}_{X \in S} \lambda, \qquad (16)$$

$$\text{at restriction } \lambda \le \lambda_k(X), k = \overline{1,K}. \qquad (17)$$

The resulting problem (16)-(17) let's call the λ-problem.

λ-problem (16)-(17) has (N+1) dimension, as a consequence of the result of the solution of λ-problem (16)-(17) represents an optimum vector of $\boldsymbol{X^o} \in R^{N+1}$, $(N+1)$ which component an essence of the value of the λ^o, i.e. $\boldsymbol{X^o} = \{x_1^o, x_2^o, \dots, x_N^o, x_{N+1}^o\}$, thus $x_{N+1}^o = \lambda^o$, and (N+1) a component of a vector of $\boldsymbol{X^o}$ selected in view of its specificity.

The received a pair of $\{\lambda^o, X^o\} = \boldsymbol{X^o}$ characterizes the optimum solution of λ-problem (16)-(17) and according to vector problem of mathematical programming (1)-(4) with the equivalent criteria, solved on the basis of normalization of criteria and the principle of the guaranteed result. We will call in the optimum solution of $\boldsymbol{X^o} = \{\lambda^o, X^o\}$, X^o - an optimal point, and λ^o - a maximum level. An important result of the algorithm for solving vector problems (1)-(4) with equivalent criteria is the following theorem.

Theorem 1. (***The theorem of the two contradictory criteria in the vector problem of mathematical programming with equivalent criteria***).

In convex vector problems of mathematical programming (1)-(4) at the equivalent criteria which is solved on the basis of normalization of criteria and the principle of the guaranteed result, in an optimum point of $\boldsymbol{X^o} = \{\lambda^o, X^o\}$ two criteria are always - denote their indexes $q \in \boldsymbol{K}, p \in \boldsymbol{K}$ (which in a sense are the most contradiction of the criteria $k = \overline{1, K}$), for which equality is carried out:

$$\lambda^o = \lambda_q(X^o) = \lambda_p(X^o), q, p \in \boldsymbol{K}, X \in \boldsymbol{S}, \tag{18}$$

and other criteria are defined by inequalities:

$$\lambda^o \leq \lambda_k(X^o), \forall k \in \boldsymbol{K},\ q \neq p \neq k. \tag{19}$$

For the first time, the proof of Theorem 1 is presented in [6, p. 22], and later it is repeated in [10, p.234]. Along with the fact that the point X^o is the optimal solution of the VPMP.

2.2.4. Mathematical algorithm of the solution of a vector problem with equivalent criteria

To solve of the vector problems of mathematical programming (1)-(4) the methods based on axiomatic of the normalization of criteria and the principle of the guaranteed result, which follow from Axiom 1 and the principle of optimality 1. The constructive method for solving a vector optimization problem with equivalent criteria includes two blocks: the 1st block "System Analysis" is divided into three steps; 2nd block "Optimal decision-making", which includes two steps: construction of the problem and its solution.

Block 1. System analysis.

Step 1. The problem (1)-(4) by each criterion separately is solved, i.e. for $\forall k \in \boldsymbol{K_1}$ is solved at the maximum, and for $\forall k \in \boldsymbol{K_2}$ is solved at a minimum. As a result of the decision, we will receive: X_k^* - an optimum point by the corresponding criterion, $k = \overline{1, K}$; $f_k^* = f_k(X_k^*)$ – the criterion size *k*th in this point, $k = \overline{1, K}$.

Step 2. We define the worst value of each criterion on $\boldsymbol{S}$: $f_k^0,\ k = \overline{1, K}$. For what the problem (1)-(4) for each criterion of $k = \overline{1, K_1}$ on a minimum is solved:

$f_k^0 = \min f_k(X), G(X) \leq B, X \geq 0, k = \overline{1, K_1}$.

The problem (1)-(4) for each criterion $k = \overline{1, K_2}$ maximum is solved:

$$f_k^0 = \max f_k(X), G(X) \leq B, X \geq 0, k = \overline{1, K_2}. \tag{20}$$

As a result of the decision, we will receive: $X_k^0 = \{x_j, j = \overline{1, N}\}$ - an optimum point by the corresponding criterion, $k = \overline{1, K}$;

$f_k^0 = f_k(X_k^0)$ – the criterion size *k*th a point, $X_k^0, k = \overline{1, K}$.

Step 3. The system analysis of a set of points, optimum across Pareto, for this purpose in optimum points of $X^* = \{X_k^*, k = \overline{1, K}\}$, are defined sizes of criterion functions of $F(X^*)$ and relative estimates $\lambda(X^*)$, $\lambda_k(X) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}, \forall k \in \boldsymbol{K}$:

$$F(X^*) = \{f_k(X_k^*), q = \overline{1, K}, k = \overline{1, K}\} = \begin{vmatrix} f_1(X_1^*), \dots, f_K(X_1^*) \\ \dots \\ f_1(X_K^*), \dots, f_K(X_K^*) \end{vmatrix}, \tag{21}$$

$$\lambda(X^*) = \{\lambda_q(X_k^*), q = \overline{1, K}, k = \overline{1, K}\} = \begin{vmatrix} \lambda_1(X_1^*), \dots, \lambda_K(X_1^*) \\ \dots \\ \lambda_1(X_K^*), \dots, \lambda_K(X_K^*) \end{vmatrix}. \tag{22}$$

As a whole on a problem $\forall k \in \boldsymbol{K}$ the relative assessment of $\lambda_k(X), k = \overline{1, K}$ (22) lies within $0 \leq \lambda_k(X) \leq 1, k = \overline{1, K}$.

Block 2. Making the optimal decision in the VPMP. It includes two steps - 4, 5.

Step 4. Creation of the λ-problem.

Creation of λ-problem is carried out in two stages:

initially built the maximine problem of optimization with the normalized criteria which at the second stage will be transformed to the standard problem of mathematical programming called λ-problem.

For construction maximine a problem of optimization we use definition 2 - relative level: $\forall X \in \boldsymbol{S} \quad \lambda = \boldsymbol{min}_{k \in \boldsymbol{K}} \lambda_k(X)$.

The bottom λ level is maximized on $X \in S$, as a result we will receive a maximine problem of optimization with the normalized criteria.

$$\lambda^o = \boldsymbol{max}_{X \in \boldsymbol{S}} \boldsymbol{min}_{k \in \boldsymbol{K}} \lambda_k(X). \tag{23}$$

At the second stage we will transform a problem (23) to a standard problem of mathematical programming:

$$\lambda^o = \boldsymbol{max}_{X \in \boldsymbol{S}}\, \lambda, \quad \rightarrow \quad \lambda^o = \boldsymbol{max}_{X \in \boldsymbol{S}}\, \lambda, \tag{24}$$

$$\lambda - \lambda_k(X) \leq 0, k = \overline{1, K}, \quad \rightarrow \quad \lambda - \frac{f_k(X) - f_k^0}{f_k^* - f_k^0} \leq 0, k = \overline{1, K}, \tag{25}$$

$$G(X) \leq B, X \geq 0, \quad \rightarrow \quad G(X) \leq B, X \geq 0, \tag{26}$$

where the vector of unknown of X has dimension of $N + 1$: $X = \{\lambda, x_1, \dots, x_N\}$.

Step 5. Solution of λ-problem.

λ-problem (24)-(26) is a standard problem of convex programming and for its decision standard methods are used. As a result of the solution of λ-problem it is received:

$$\boldsymbol{X^o} = \{\boldsymbol{X^o}, \boldsymbol{\lambda^o}\} \text{ - an optimum point;} \tag{27}$$

$$f_k(X^o), k = \overline{1, K} \text{ are values of the criteria in this point;} \tag{28}$$

$$\lambda_k(X^o) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K} \text{ are sizes of relative estimates;} \tag{29}$$

λ^o - the maximum relative estimates which is the maximum bottom level for all relative estimates of $\lambda_k(X^o)$, or the guaranteed result in relative units. λ^o guarantees that all relative estimates of $\lambda_k(X^o)$ more or are equal λ^o :

$$\lambda_k(X^o) \geq \boldsymbol{\lambda^o}, k = \overline{1, K} \text{ or } \boldsymbol{\lambda^o} \leq \lambda_k(X^o), k = \overline{1, K}, X^o \in \boldsymbol{S}, \tag{30}$$

and according to the theorem 1 point of $X^o = \{\boldsymbol{\lambda^o}, x_1, \dots, x_N\}$ is optimum across Pareto.

2.3. The axioms and the principle of optimality for vector optimization with a Criterion Priority

Definition 3 states that if we compare two functions (criteria) measured in relative estimates $\lambda_{k=1}(X)$ and $\lambda_{k=2}(X)\ \forall k \in K$, then three situations are possible. The second situation, when $\lambda_{k=1}(X) = \lambda_{k=2}(X)$ is investigated in Section 2.2 (equivalent criteria). Situations: the first, when $\lambda_{k=1}(X) > \lambda_{k=2}(X)$, and the third, when $\lambda_{k=1}(X) < \lambda_{k=2}(X)$, are explored in this section. Such situations are defined as tasks with the priority of the criterion.

For development of methods of the solution of problems of vector optimization with a priority of criterion we use definitions as follows: Priority of one criterion of vector problems, with a criterion priority over other criteria; Numerical expression of a priority; The set priority of a criterion; the lower (minimum) level from all criteria with a priority of one of them; a subset of points with priority by criterion (Axiom 2); the principle of optimality of the solution of problems of vector optimization with the set priority of one of the criteria, and related theorems. For more details see [20, 22].

2.3.1. Axiomatics of solving a Vector Optimization Problem with a given criterion priority

The language of the axiomatics system for solving a vector problem with a given criterion priority includes definitions: 1) Priority of one criterion over another; 2) The numerical value of the priority of the criterion; 3) The lowest level of the criterion among all relative evaluations with the priority of the criterion.

Definition 6. (***About the priority of one criterion over the other***).

The criterion of $q \in \boldsymbol{K}$ in the vector problem of Equations (1)-(4) in a point of $X \in \boldsymbol{S}$ has priority over other criteria of $k = \overline{1, K}$, and the relative estimate of $\lambda_q(X)$ by this criterion is greater than or equal to relative estimates of $\lambda_k(X)$ of other criteria, i.e.:

$$\lambda_q(X) \geq \lambda_k(X), k = \overline{1, K}, \tag{31}$$

and a strict priority for at least one criterion of $t \in \boldsymbol{K}$,

$\lambda_q(X) > \lambda_k(X), t \neq q$, and for other criteria of $\lambda_q(X) \geq \lambda_k(X), k = \overline{1, K}, k \neq t \neq q$.

Introduction of the definition of a priority of criterion $q \in \boldsymbol{K}$ in the vector problem of Equations (1)-(4) executed the redefinition of the early concept of a priority. Earlier the intuitive concept of the importance of this criterion was outlined, now this "importance" is defined as a mathematical concept: the higher the relative estimate of the qth criterion compared to others, the more it is important (i.e., more priority), and the highest priority at a point of an optimum is $X_k^*, \forall q \in \boldsymbol{K}$.

From the definition of a priority of criterion of $q \in \boldsymbol{K}$ in the vector problem of Equations (1)-(4), it follows that it is possible to reveal a set of points $\boldsymbol{S}_q \subset \boldsymbol{S}$ that is characterized by

$\lambda_q(X) \geq \lambda_k(X), \forall k \neq q, \forall X \in \boldsymbol{S}_q$. However, the answer to whether a criterion of $q \in \boldsymbol{K}$ at a point of the set $\boldsymbol{S}_q$ has more priority than others do remains open. For clarification of this question, we define a communication coefficient between a couple of relative estimates of q and k that, in total, represent a vector:

$P^q(X) = \{p_k^q(X) \mid k = \overline{1, K}\}, q \in \boldsymbol{K}\ \forall X \in \boldsymbol{S}_q$.

Definition 7. (***About numerical expression of a priority of one criterion over another***).

In the vector problem of Equations (1) and (4), with priority of the qth criterion over other criteria of $k = \overline{1, K}$, for $\forall X \in \boldsymbol{S}_q$, and a vector of $P^q(X)$ which shows how many times a relative estimate of $\lambda_q(X), q \in \boldsymbol{K}$, is more than other relative estimates of $\lambda_k(X), k = \overline{1, K}$, we define a numerical expression of the priority of the qth criterion over other criteria of $k = \overline{1, K}$ as:

$$P^q(X) = \{p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}, k = \overline{1, K}\}, p_k^q(X) \geq 1, \forall X \in \boldsymbol{S}_q \subset S, k = \overline{1, K}, \forall q \in \boldsymbol{K}. \tag{32}$$

Такое отношение $p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}$ назовем *числовым выражением приоритета q-го критерия над остальными критериями* $k = \overline{1, K}$. Such a ratio $p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}$. let us call the numerical expression of the priority of the q-th criterion over the rest of the criteria $k = \overline{1, K}$.

Definition 7a. (***On a given numerical expression of the priority of one criterion over others***). In the vector problem of Equations (1)–(4) with a priority of criterion of $q \in \boldsymbol{K}$ for $\forall X \in \boldsymbol{S}$, vector $P^q = \{p_k^q, k = \overline{1, K}\}$ is considered to be set by the person making decisions (i.e., decision-maker) if everyone is set a component of this vector. Set by the decision-maker, componentp_k^q, from the point of view of the decision-maker, shows how many times a relative estimate of $\lambda_q(X), q \in \boldsymbol{K}$ is greater than other relative estimates of $\lambda_k(X), k = \overline{1, K}$. The vector of $p_k^q, k = \overline{1, K}$, is the numerical expression of the priority of the qth criterion over other criteria of $k = \overline{1, K}$:

$$P^q(X) = \{p_k^q(X), k = \overline{1, K}\}, p_k^q(X) \geq 1, \forall X \in \boldsymbol{S}_q \subset S, k = \overline{1, K}, \forall q \in \boldsymbol{K}. \tag{33}$$

The vector problem of Equations (1)–(4), in which the priority of any criteria is set, is called a vector problem with the set priority of criterion. The problem of a task of a vector of priorities arises when it is necessary to determine the point $X^o \in \boldsymbol{S}$ by the set vector of priorities. In the comparison of relative estimates with a priority of criterion of $q \in \boldsymbol{K}$, as well as in a task with equivalent criteria, we define the additional numerical characteristic of λ which we call the level.

Definition 8. (***About the lower level among all relative estimates with a criterion priority***).

The λ level is the lowest among all relative estimates with a priority of criterion of $q \in \boldsymbol{K}$ such that:

$$\lambda \leq p_k^q \lambda_k(X),\ k = \overline{1, K},\ q \in \boldsymbol{K},\ \forall X \in \boldsymbol{S}_q \subset S; \tag{34}$$

The lower level for the performance of the condition in Equation (34) is defined as:

$$\lambda = min_{k \in K} p_k^q \lambda_k(X), q \in \boldsymbol{K}, \forall X \in \boldsymbol{S}_q \subset S. \tag{35}$$

Equations (34) and (35) are interconnected and serve as a further transition from the operation of the definition of the minimum to restrictions, and vice versa. In Section 4, we gave the definition of a Pareto optimal point $X^o \in \boldsymbol{S}$ with equivalent criteria. Considering this definition as an initial one, we will construct a number of the axioms dividing an admissible set of $\boldsymbol{S}$ into, first, a subset of Pareto optimal points $S°$, and, secondly, a subset of points $\boldsymbol{S}_q \subset \boldsymbol{S}, q \in \boldsymbol{K}$, with priority for the qth criterion.

2.3.2. The axioms for vector optimization with a Criterion Priority

Axiom 2. (About a subset of points, priority by criterion).

In the vector problem of Equations (1)–(4), the subset of points $\boldsymbol{S}_q \subset \boldsymbol{S}$ is called the area of priority of criterion of $q \in \boldsymbol{K}$ over other criteria, if

$\forall X \in \boldsymbol{S}_q \ \forall k \in \boldsymbol{K} \ \lambda_q(X) \geq \lambda_k(X), q \neq k.$

This definition extends to a set of Pareto optimal points $\boldsymbol{S}^o$ that is given by the following definition.

Axiom 2a. (*About a subset of points, priority by criterion*, on Pareto's great number in a vector problem). In a vector problem of mathematical programming the subset of points $\boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset$ S is called the area of a priority of criterion of $q \in \boldsymbol{K}$ over other criteria, if

$\forall \mathrm{X} \in S_q^o \ \forall k \in K \ \lambda_q(X) \geq \lambda_k(X), q \neq k.$

In the following we provide explanations.

Axiom 2 and 2a allow the breaking of the vector problem in Equations (1)–(4) into an admissible set of points $\boldsymbol{S}$, including a subset of Pareto optimal points, $\boldsymbol{S}^o \subset \boldsymbol{S}$, and subsets:

One subset of points $\boldsymbol{S}' \subset \boldsymbol{S}$ where criteria are equivalent, and a subset of points of $\boldsymbol{S}'$ crossed with a subset of points $\boldsymbol{S}°$, allocated to a subset of Pareto optimal points at equivalent criteria $\boldsymbol{S}^{oo} = \boldsymbol{S}' \cap \boldsymbol{S}^o$. As will be shown further, this consists of one point of $X^o \in \boldsymbol{S}$, i.e.

$X^o = \boldsymbol{S}^{oo} = \boldsymbol{S}' \cap \boldsymbol{S}^o, \boldsymbol{S}' \in \boldsymbol{S}, \boldsymbol{S}^o \subset \boldsymbol{S}.$

"$\boldsymbol{K}$" subsets of points where each criterion of $q = \overline{1, K}$ has a priority over other criteria of $k = \overline{1, K}, q \neq k,$ and thus breaks, first, sets of all admissible points $\boldsymbol{S}$, into subsets $\boldsymbol{S}_q \subset \boldsymbol{S}, q = \overline{1, K}$ and, second, a set of Pareto optimal points, $\boldsymbol{S}°$, into subsets $\boldsymbol{S}_q^o \subset \boldsymbol{S}_q \subset \boldsymbol{S}, q = \overline{1, K}$. This yields:

$(\boldsymbol{S}' \cup (\bigcup_{q \in K} \boldsymbol{S}_q^o) \equiv \boldsymbol{S}°, \boldsymbol{S}_q^o \subset \boldsymbol{S}° \subset \boldsymbol{S}, q = \overline{1, K}.$

We note that the subset of points $\boldsymbol{S}_q^o$, on the one hand, is included in the area (a subset of points) of priority of criterion of $q \in \boldsymbol{K}$ over other criteria: $\boldsymbol{S}_q^o \subset \boldsymbol{S}_q \subset \boldsymbol{S}$, and, on the other, in a subset of Pareto optimal points $\boldsymbol{S}_q^o \subset \boldsymbol{S}° \subset \boldsymbol{S}$.

Axiom 2 and the numerical expression of priority of criterion (Definition 5) allow the identification of each admissible point of $X \in \boldsymbol{S}$ (by means of vector:

$P^q(X) = \{p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}, k = \overline{1, K}\}$, to form and choose:

a subset of points by priority criterion $\boldsymbol{S}_q$, which is included in a set of points $\boldsymbol{S}$, $\forall q \in \boldsymbol{K} \ X \in \boldsymbol{S}_q \subset \boldsymbol{S}$, (such a subset of points can be used in problems of clustering, but is beyond this article);

a subset of points by priority criterion $\boldsymbol{S}_q^o$, which is included in a set of Pareto optimal points $\boldsymbol{S}°$, $\forall q \in K, X \in \boldsymbol{S}_q^o \subset \boldsymbol{S}°$.

Thus, full identification of all points in the vector problem of Equations (1)–(4) is executed in sequence as:

Set of admissible points of $X \in \boldsymbol{S}$ →	Subset of points, opti mum across Pareto, $X \in \boldsymbol{S}^o \subset \boldsymbol{S}$ →	Subset of points, optimum across Pareto $X \in \boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset \boldsymbol{S}$ →	Separate point of a $\forall X \in \boldsymbol{S}$ $X \in \boldsymbol{S}_q^o \subset \boldsymbol{S}^o \subset \boldsymbol{S}$

This is the most important result which allows the output of the principle of optimality and to construct methods of a choice of any point of Pareto's great number.

2.3.3. The principle of optimality for vector optimization with a Criterion Priority

Definition 9. (***Principle of optimality 2. The solution of a vector problem with the set criterion priority***).

The vector problem of Equations (1)–(4) with the set priority of the qth criterion of $p_k^q\lambda_k(X), k = \overline{1,K}$ is considered solved if the point $X°$ and maximum level $\lambda°$ among all relative estimates is found such that:

$$\lambda^o = \boldsymbol{max}_{X\in S}\boldsymbol{min}_{k\in K}\, p_k^q\lambda_k(X), q \in \boldsymbol{K}. \quad (36)$$

Using the interrelation of Equations (34) and (35), we can transform the maximine problem of Equation (36) into an extreme problem of the form:

$$\lambda^o = \boldsymbol{max}_{X\in S}\lambda, \quad (37)$$

$$\text{at restriction } \lambda \le p_k^q\lambda_k(X), k = \overline{1,K}. \quad (38)$$

We call Equations (37) and (38) the λ-problem with a priority of the qth criterion.

The solution of the λ-problem is the point $\boldsymbol{X^o} = \{X^o, \lambda^o\}$. This is also the result of the solution of the vector problem of Equations (1)-(4) with the set priority of the criterion, solved on the basis of normalization of criteria and the principle of the guaranteed result.

In the optimum solution $\boldsymbol{X^o} = \{X^o, \lambda^o\}$, $X°$, an optimum point, and $\lambda°$, the maximum bottom level, the point of $X°$ and the $\lambda°$ level correspond to restrictions of Equation (4), which can be written as: $\lambda^o \le p_k^q\lambda_k(X^o), k = \overline{1,K}$.

These restrictions are the basis of an assessment of the correctness of the results of a decision in practical vector problems of optimization.

From Definitions 1 and 2, "Principles of optimality", follows the opportunity to formulate the concept of the operation “opt”.

Definition 9. (***Mathematical operation "opt"***).

In the vector problem of Equations (1)–(4), in which "max" and "min" are part of the criteria, the mathematical operation "opt" consists of the definition of a point $X°$ and the maximum $\lambda°$ bottom level to which all criteria measured in relative units are lifted:

$$\lambda° \le \lambda_k(X°) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}, k = \overline{1,K}, \quad (39)$$

i.e., all criteria of $\lambda_k(X^o), k = \overline{1,K}$, are equal to or greater than the maximum level of $\lambda°$ (therefore $\lambda°$ is also called the guaranteed result).

Theorem 2. (*The theorem of the most inconsistent criteria in a vector problem with the set priority*).

If in the convex vector problem of mathematical programming of Equations (1)–(4) the priority of the qth criterion of $p_k^q, k = \overline{1,K}, \forall q \in \boldsymbol{K}$ over other criteria is set, at a point of an optimum $X^o \in \boldsymbol{S}$ obtained on the basis of normalization of criteria and the principle of guaranteed result, there will always be two criteria with the indexes $r \in \boldsymbol{K},\ t \in \boldsymbol{K}$, for which the following strict equality holds:

$$\lambda° = p_k^r\lambda_r(X°) = p_k^t\,\lambda_t(X°), r, t, \in \boldsymbol{K}, \quad (40)$$

and other criteria are defined by inequalities:

$$\lambda° \le p_k^q(X°), k = \overline{1,K}, \forall q \in \boldsymbol{K}, q \ne r \ne t\ . \quad (41)$$

Criteria with the indexes $r \in \boldsymbol{K},\ t \in \boldsymbol{K}$ for which the equality of Equation (41) holds are called the most inconsistent.

Proof. Similar to Theorem 2 [20].

We note that in Equations (40) and (41), the indexes of criteria $r, t \in \boldsymbol{K}$ can coincide with the $q \in \boldsymbol{K}$ index.

Consequence of Theorem 1, about equality of an optimum level and relative estimates in a vector problem with two criteria with a priority of one of them.

In a convex vector problem of mathematical programming with two equivalent criteria, solved on the basis of normalization of criteria and the principle of the guaranteed result, at an optimum point X^o equality is always carried out at a priority of the first criterion over the second:

$$\lambda^o = \lambda_1(X^o) = p_2^1(X^o)\lambda_2(X^o), X^o \in \boldsymbol{S}\,, \tag{42}$$

where $p_2^1(X^o) = \lambda_1(X^o)/\lambda_2(X^o)$,

and at a priority of the second criterion over the first:

$\lambda^o = \lambda_2(X^o) = p_1^2(X^o)\lambda_1(X^o), X^o \in \boldsymbol{S}$, where $p_1^2(X^o) = \lambda_2(X^o)/\lambda_1(X^o)$.

2.3.4. Mathematical Method of the Solution of a Vector Problem with Criterion Priority

Step 1. We solve a vector problem with equivalent criteria. The algorithm of the decision is presented in Section 4.4.

As a result of the decision, we obtain:

optimum points by each criterion separately $X_k^*, k = \overline{1,K}$ and sizes of criterion functions in these points of $f_k^* = f_k(X_k^*), k = \overline{1,K}$, which represent the boundary of a set of Pareto optimal points;

anti-optimum points by each criterion of $X_k^0 = \{x_j, j = \overline{1,N}\}$ and the worst unchangeable part of each criterion of $f_k^0 = f_k(X_k^0), k = \overline{1,K}$;

$\boldsymbol{X^o} = \{X^o, \lambda^o\}$, an optimum point, as a result of the solution of VPMP at equivalent criteria, i.e., the result of the solution of a maximine problem and the λ-problem constructed on its basis;

λ°, the maximum relative assessment which is the maximum lower level for all relative estimates of $\lambda_k(X^o)$, or the guaranteed result in relative units, λ° guarantees that all relative estimates of $\lambda_k(X^o)$ are equal to or greater than λ°:

$$\lambda^o \le \lambda_k(X^o), k = \overline{1,K}, X^o \in S\,. \tag{43}$$

The person making the decision carries out the analysis of the results of the solution of the vector problem with equivalent criteria.

If the received results satisfy the decision maker, then the process concludes, otherwise subsequent calculations are performed.

In addition, we calculate:

in each point $X_k^*, k = \overline{1,K}$ we determine sizes of all criteria of: $q = \overline{1,K}$

$\{f_q(X_k^*), q = \overline{1,K}\}, k = \overline{1,K}$, and relative estimates

$\lambda(X^*) = \{\lambda_q(X_k^*), q = \overline{1,K}, k = \overline{1,K}\},\ \lambda_k(X) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}, \forall k \in \boldsymbol{K}$:

$$F(X^*) = \begin{vmatrix} f_1(X_1^*) \dots f_K(X_1^*) \\ \dots \\ f_1(X_K^*) \dots f_K(X_K^*) \end{vmatrix},\ \lambda(X^*) = \begin{vmatrix} \lambda_1(X_1^*) \dots \lambda_K(X_1^*) \\ \dots \\ \lambda_1(X_K^*) \dots \lambda_K(X_K^*) \end{vmatrix}. \tag{44}$$

Matrices of criteria of $F(X^*)$ and relative estimates of $\lambda(X^*)$ show the sizes of each criterion of $k = \overline{1,K}$ upon transition from one optimum point $X_k^*, k \in \boldsymbol{K}$ to another $X_q^*, q \in \boldsymbol{K}$, i.e., on the border of a great number of Pareto.

at an optimum point at equivalent criteria X^o we calculate sizes of criteria and relative estimates:

$$f_k(X^o), k = \overline{1,K}; \lambda_k(X^o), k = \overline{1,K}, \tag{45}$$

which satisfy the inequality of Equation (43). In other points $X \in \boldsymbol{S^o}$, in relative units the criteria of $\lambda = \boldsymbol{min}_{k \in K}\lambda_k(X)$ are always less than λ°, given the λ-problem of Equations (24)-(26). This information is also a basis for further study of the structure of a great number of Pareto.

Step 2. Choice of priority criterion of $q \in \boldsymbol{K}$.

From theory (see Theorem 1) it is known that at an optimum point X^o there are always two most inconsistent criteria, $q \in \boldsymbol{K}$ and $v \in \boldsymbol{K}$, for which in relative units an exact equality holds:

$\lambda^o = \lambda_q(X^o) = \lambda_v(X^o), q, v \in \boldsymbol{K}, X \in \boldsymbol{S}$. Others are subject to inequalities:

$\lambda^o \le \lambda_k(X^o), \forall k \in \boldsymbol{K},\ q \ne v \ne k$.

As a rule, the criterion which the decision-maker would like to improve is part of this couple, and such a criterion is called a priority criterion, which we designate $q \in \boldsymbol{K}$.

Step 3. Numerical limits of the change of the size of a priority of criterion $q \in \boldsymbol{K}$ are defined.

For priority criterion $q \in \boldsymbol{K}$ from the matrix of Equation (40) we define the numerical limits of the change of the size of criterion:

in physical units of $f_k(X^o) \leq f_q(X) \leq f_q(X_q^*), k \in \boldsymbol{K}$, (46)

where $f_q(X_q^*)$ derives from the matrix of Equation $F(X^*)$ (44), all criteria showing sizes measured in physical units, $f_k(X^o), k = \overline{1, K}$ from Equation (45), and,

in relative units of $\lambda_k(X^o) \leq \lambda_q(X) \leq \lambda_q(X_q^*), k \in \boldsymbol{K}$, (47)

where $\lambda_q(X_q^*)$derives from the matrix $\lambda(X^*)$, all criteria showing sizes measured in relative units (we note that $\lambda_q(X_q^*) = 1), \lambda_q(X^o)$ from Equation (44).

As a rule, Equations (46) and (47) are given for the display of the analysis.

Step 4. Choice of the size of priority criterion (decision-making).

The person making the decision carries out the analysis of the results of calculations of Equation 5.14) and from the inequality of Equation (46) chooses the numerical size f_q of the criterion of $q \in \boldsymbol{K}$:

$$f_q(X^o) \leq f_q \leq f_q(X_q^*), q \in \boldsymbol{K}. \quad (48)$$

For the chosen size of the criterion of f_q it is necessary to define a vector of unknown X^{oo}. For this purpose, we carry out the subsequent calculations.

Step 5. Calculation of a relative assessment.

For the chosen size of the priority criterion of f_q the relative assessment is calculated as:

$$\lambda_q = \frac{f_q - f_q^0}{f_q^* - f_q^0},$$

which upon transition from point X^o to X_q^*, according to Equation (43), lies in the limits:

$$\lambda_q(X^o) \leq \lambda_q \leq \lambda_q(X_q^*) = 1.$$

Step 6. Calculation of the coefficient of linear approximation.

Assuming a linear nature of the change of criterion of $f_q(X)$ in Equation (48) and according to the relative assessment of $\lambda_q(X)$, using standard methods of linear approximation we calculate the proportionality coefficient between$\lambda_q(X^o), \lambda_q$, which we call ρ:

$$\rho = \frac{\lambda_q - \lambda_q(X^o)}{\lambda_q^* - \lambda_q^0}, q \in \boldsymbol{K}.$$

Step 7. Calculation of coordinates of priority criterion with the size f_q.

In accordance with Equation (44), the coordinates of the X_q priority criterion point lie within the following limits: $X^o \leq X_q \leq X_q^*, q \in \boldsymbol{K}$. Assuming a linear nature of change of the vector

$X_q = \{x_1^q, \dots, x_N^q\}$ we determine coordinates of a point of priority criterion with the size f_q with the relative assessment of Equation (45):

$$X_q = \{x_1^q = x_1^o + \rho(x_q^*(1) - x_1^o), \dots,$$
$$x_N^q = x_N^o + \rho(x_q^*(N) - x_N^o)\}, \quad (49)$$

where $X^o = \{x_1^o, \dots, x_N^o\}, X_q^* = \{x_q^*(1), \dots, x_q^*(N)\}$.

Step 8. Calculation of the main indicators of a point x_q (49).

For the obtained point x_q, we calculate:

all criteria in physical units: $F^q = \{f_k(x^q), k = \overline{1, K}\}$;

all relative estimates of criteria:

$$\lambda^q = \{\lambda_k^q, k = \overline{1, K}\}, \lambda_k(x^q) = \frac{f_k(x^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}; \quad (50)$$

the vector of priorities: $P^q = \{p_k^q = \frac{\lambda_q(x^q)}{\lambda_k(x^q)}, k = \overline{1, K}\}$;

the maximum relative assessment: $\lambda^{oq} = \min(p_k^q \lambda_k(x^q), k = \overline{1, K})$.

Any (50) point from Pareto's set $\boldsymbol{X}_t^o = \{\lambda_t^o, X_t^o\} \in \boldsymbol{S}^o$ can be similarly calculated.

Analysis of results. The calculated size of criterion $f_q(X_t^o), q \in \boldsymbol{K}$ is usually not equal to the set f_q. The error of the choice of $\Delta f_q = |f_q(X_t^o) - f_q|$ is defined by the error of linear approximation. The results of the study of *symmetry* in VPMP with a given priority are similar as for VPMP with equivalent criteria, but the center of symmetry is shifted towards the priority criterion.

2.4. Applied Multidimensional Mathematics: Vector Problem of Nonlinear Programming as a Model of Engineering Systems Development

In this section, we consider a separate problem of multidimensional mathematics: the vector nonlinear programming problem. Mathematical and software for solving a vector problem of nonlinear programming, the algorithms for solving which with equivalent criteria are presented in Section 3 and with a given priority of the criterion in Section 4, [20, 22, 43, 45].

2.4.1. Vector Problem of Nonlinear Programming

The vector problem of nonlinear programming (VPNP) is a standard problem of mathematical programming, which has some set of criteria that together represent a vector of criteria. VPNP are divided into homogeneous and heterogeneous VPNPs. In accordance with these definitions, we will present a vector problem of nonlinear programming with heterogeneous criteria.

$$Opt\ F(X) = \{\max F_1(X) = \{\max f_k(X) = c_{0k} + c_{1k}x_1 + \cdots + c_{N+1,k}x_1x_2 + \cdots + c_{n1,k}x_1^2 + \cdots + c_{nnk}x_n^2, k = \overline{1, K_1}\}, \quad (51)$$

$$\min F_2(X) = \{\min f_k(X) = c_{0k} + c_{1k}x_1 + \cdots + c_{N+1,k}x_1x_2 + \cdots + c_{n1,k}x_1^2 + \ldots + c_{nnk}x_n^2, k = \overline{1, K_2}\}\}, \quad (52)$$

$$a_{0k} + a_{1i}x_1 + \cdots + a_{N+1,i}x_1x_2 + \cdots + a_{n1,i}x_1^2 + \cdots + a_{nni}x_n^2 \leq b_i,\ \ i = \overline{1, M}, \quad (53)$$

$$0 \leq x_j(t) \leq u_j\ , j = \overline{1, N}, \quad (54)$$

where $X = \{x_j, j = \overline{1, N}\}$ is a vector of variables, i.e. it is a vector of the N-dimensional Euclidean space R^N, (the notation $j = \overline{1, N}$ is equivalent to $j = (1, \ldots, N)$;

$F(X)$ is a vector function (vector criterion) that has K is a component of functions. The function represents a quadratic polynomial. The set of criteria (polynomials) K consists of a subset of the K_1 component of maximization and a subset of the K_2 minimization: $K = K_1 \cup K_2$, to evaluate the set of criteria, the designation "*opt*" operation is introduced, which includes max and min;

$F_1(X) = \{max\ f_k(X), k = \overline{1, K_1}$ is a vector criterion (51), each component of which maximizes, K_1 is the number of criteria, and the $K_1 \equiv \overline{1, K_1}$ is the set of maximization criteria (task (51), (53)-(54) are VPNPs with homogeneous maximization criteria);

$F_2(X) = \{min\ f_k(X), k = \overline{1, K_2}$ is a vector criterion, each component of which is minimized, $K_2 \equiv \overline{K_1 + 1, K} \equiv \overline{1, K_2}$ is a set of minimization criteria, K_2 is the number of criteria, (task (52),(53)-(54) is an VPNP with homogeneous minimization criteria). $K_1 \cup K_2 = K$, $K_1 \subset K$, $K_2 \subset K$.

(53) - (54) - standard nonlinear constraints (in the form of polynomials, including linear ones).

$S = \{X \in R^n | G(X) \leq 0, X^{min} \leq X \leq X^{max}\} \neq \emptyset$

is the allowable set of points (or more briefly, the allowable set) given by the standard constraints (53)-(54) and the trivial constraints $0 \leq x_j(t) \leq u_j, j = \overline{1, N}$. Suppose that the allowable set of points is not empty and is compact. The vector function (criterion) of minimization $F_2(X)$ can be transformed into a vector function (criterion) of maximization by multiplying each component of $F_2(X)$ by minus one. Vector criterion $F_2(X)$ is introduced in the VPNP (51)-(54) for in order to show that the problem has two subsets of criteria $\boldsymbol{K_1}, \boldsymbol{K_2}$ with fundamentally different directions of optimization, [46].

2.4.2. Software Structure for Solving a Vector Problem of Nonlinear Programming

The software for solving the vector problem of nonlinear programming (51)-(54) is implemented on the basis of the algorithm for solving the VPNP and the use of the FMINCON program (...) in the MATLAB system, [20, 21, 22]. When using the FMINCON(...) program, it is necessary to develop two subroutines – functions.

The first function includes two blocks: the first block is designed to evaluate the criterion $f_k(X)\ \forall k \in K$ at point *X*; the second block to calculate the first derivative at this point $\frac{df_k(X)}{dX}\ \forall k \in K$.

The second function includes the same two blocks for constraints only: $g_i(X)\ \forall i \in M$ and $\frac{dg_i(X)}{dX}\ \forall i \in M$.

The FMINCON(...) program is used in the first step of the algorithm (maximizing the criteria) and in the second step of the algorithm (minimization). Similarly, in accordance with the algorithm, step 4 and 5 are solved λ-problem.

In general, under nonlinear constraints, the software of the VPNP solution includes:

*K** 2 (1 step) + *K* * 2 (2 step) + 2 (λ-problem) functions. Since the criteria and limitations of the VPNP are individual, individual software is written for each VPNP. *Numerical implementation of a vector problem of nonlinear programming*

Example.

It is given. The consideration of the vector nonlinear (convex) programming problem with four homogeneous criteria. In terms of criteria, we use a circle, and with the linear restrictions the problem is therefore solved orally and imposed on variables.

$$opt\ F(X) = \{min\ F_2(X) = \{min\ f_1(X) \equiv (x_1 - 80)^2 + (x_2 - 80)^2,$$

$$min\ f_2(X) \equiv (x_1 - 80)^2 + (x_2 - 20)^2,$$

$$min\ f_3(X) \equiv (x_1 - 20)^2 + (x_2 - 20)^2,$$

$$min\ f_4(X) \equiv (x_1 - 20)^2 + (x_2 - 80)^2\},$$

at restrictions $0 \le x_1 \le 100,\ 0 \le x_2 \le 100.$

Need to be determined. Develop software in MATLAB solutions vector problem nonlinear programming. Using software solve the problem (5.1)-(5.5).

Software for solving the vector problem of nonlinear programming (VPNP)

To solve the vector problem of nonlinear programming - a model of an engineering system, a program has been developed in the MATLAB system, which implements an algorithm for solving VPNP with equivalent criteria. The following is the result of the VPNP decision obtained by the program.

Recording the program in MATLAB format

```
% Программа "Решение векторной задачи нелинейного программирования":
 function [x,f] = VPNP_2_4Krit_100(x)
% Автор: Машунин Юрий Константинович (Mashunun Yu. K.)
% Алгоритм и программа предназначена для использования в образовании и научных
% исследованиях, для коммерческого использования обращаться: Mashunin@mail.ru
% Algorithm VPNP: 4Kritery + L-zadaha
%[X,Fval,EXITFLAG,OUTPUT,LAMBDA,GRAD,HESSIAN]=
%                                                      FMINCON(FUN,Xo,A,b,Aeq,beq,
lb,ub,nonlcon,options,P1,P2,...)
disp('*** Блок Исходных данных. ВЗНП:***')
disp('opt F(X)={max F1(X)={min f1=(x1-80).^2+(x2-80).^2; ')
disp('                        min f2=(x1-80).^2+(x2-20).^2; ')
disp('                        min f3=(x1-20).^2+(x2-20).^2; ')
disp('                        min f4=(x1-20).^2+(x2-80).^2; ')
disp('                                 0<=x1<=100, 0<=x2<=100 ')
lb=[0. 0.];
ub=[100. 100.]; Xo=[0. 0.];
options=optimset('LargeScale','off');
options=optimset(options,'GradObj','on','GradConst','off');
```

```
A=[1 0;
   0 1];
b=[100 100];
Aeq=[]; beq=[];
XoK1max=[0. 0.];
disp('*** Шаг 1. Решение по каждому критерию (наилучшее) ***')%
[x1max,f1max]= fmincon('VPNP_2_Krit1max',XoK1max,A,b,Aeq,beq,lb,ub,'',options)
[f1X1max] = VPNP_2_Krit1min(x1max)
[f2X1max] = VPNP_2_Krit2min(x1max)
[f3X1max] = VPNP_2_Krit3min(x1max)
[f4X1max] = VPNP_2_Krit4min(x1max)
XoK2max=[0. 0.];
[x2max,f2max]= fmincon('VPNP_2_Krit2max',XoK2max,A,b,Aeq,beq,lb,ub,'',options)
[f1X2max] = VPNP_2_Krit1min(x2max)
[f2X2max] = VPNP_2_Krit2min(x2max)
[f3X2max] = VPNP_2_Krit3min(x2max)
[f4X2max] = VPNP_2_Krit4min(x2max)
XoK3max=[0. 0.];
[x3max,f3max]= fmincon('VPNP_2_Krit3max',XoK3max,A,b,Aeq,beq,lb,ub,'',options)
[f1X3max] = VPNP_2_Krit1min(x3max)
[f2X3max] = VPNP_2_Krit2min(x3max)
[f3X3max] = VPNP_2_Krit3min(x3max)
[f4X3max] = VPNP_2_Krit4min(x3max)
XoK4max=[0. 0.];
[x4max,f4max]= fmincon('VPNP_2_Krit4max',XoK4max,A,b,Aeq,beq,lb,ub,'',options)
[f1X4max] = VPNP_2_Krit1min(x4max)
[f2X4max] = VPNP_2_Krit2min(x4max)
[f3X4max] = VPNP_2_Krit3min(x4max)
[f4X4max] = VPNP_2_Krit4min(x4max)
disp('*** Шаг 2. Решение по каждому критерию (наихудшее) ***')%
XoK1min=[0. 0.];
[x1min,f1min]= fmincon('VPNP_2_Krit1min',XoK1min,A,b,Aeq,beq,lb,ub,'',options)
[f1X1min] = VPNP_2_Krit1min(x1min)
[f2X1min] = VPNP_2_Krit2min(x1min)
[f3X1min] = VPNP_2_Krit3min(x1min)
[f4X1min] = VPNP_2_Krit4min(x1min)
[x2min,f2min] = fmincon('VPNP_2_Krit2min',Xo,A,b,Aeq,beq,lb,ub,'',options)
[f1X2min] = VPNP_2_Krit1min(x2min)
[f2X2min] = VPNP_2_Krit2min(x2min)
[f3X2min] = VPNP_2_Krit3min(x2min)
[f4X2min] = VPNP_2_Krit4min(x2min)
[x3min,f3min] = fmincon('VPNP_2_Krit3min',Xo,A,b,Aeq,beq,lb,ub,'',options)
[f1X3min] = VPNP_2_Krit1min(x3min)
[f2X3min] = VPNP_2_Krit2min(x3min)
[f3X3min] = VPNP_2_Krit3min(x3min)
[f4X3min] = VPNP_2_Krit4min(x3min)
[x4min,f4min] = fmincon('VPNP_2_Krit4min',Xo,A,b,Aeq,beq,lb,ub,'',options)
[f1X4min] = VPNP_2_Krit1min(x4min)
[f2X4min] = VPNP_2_Krit2min(x4min)
[f3X4min] = VPNP_2_Krit3min(x4min)
[f4X4min] = VPNP_2_Krit4min(x4min)
disp('*** Шаг 3. Системный анализ результатов ***')%
disp('Оценка критериев в точках оптимума: X1min,X2min,X3min,X4min')%
F=[f1X1min f2X1min f3X1min f4X1min;
   f1X2min f2X2min f3X2min f4X2min;
   f1X3min f2X3min f3X3min f4X3min;
   f1X4min f2X4min f3X4min f4X4min]
d1=f1X1min-f1X1max
d2=f2X2min-f2X2max
```

```
d3=f3X3min-f3X3max
d4=f4X4min-f4X4max
disp('Оценка критериев в относительных единицах: X1min,X2min,X3min,X4min')%
L=[(f1X1min-f1X1max)/d1   (f2X1min-f2X2max)/d2   (f3X1min-f3X3max)/d3   (f4X1min-f4X4max)/d4;
   (f1X2min-f1X1max)/d1   (f2X2min-f2X2max)/d2   (f3X2min-f3X3max)/d3   (f4X2min-f4X4max)/d4;
   (f1X3min-f1X1max)/d1   (f2X3min-f2X2max)/d2   (f3X3min-f3X3max)/d3   (f4X3min-f4X4max)/d4;
   (f1X4min-f1X1max)/d1   (f2X4min-f2X2max)/d2   (f3X4min-f3X3max)/d3   (f4X4min-f4X4max)/d4]
disp('*** Шаг 4. Построение L-задачи ***')%
Ao=[1 0 0;
    0 1 0];
bo=[100 100]; Aeq=[]; beq=[];
Xoo=[0 0 0]
lbo=[0. 0. 0.]
ubo=[100. 100. 1]
disp('*** Шаг 5. Решение L-задачи ***')%
[Xo,Lo]=fmincon('VPNP_2_L',Xoo,Ao,bo,Aeq,beq,lbo,ubo,'VPNP_2_LConst') %,options)
disp('Оценка критериев в точке оптимума Xo')%
[f1Xo] = VPNP_2_Krit1min(Xo(1:2))
[f2Xo] = VPNP_2_Krit2min(Xo(1:2))
[f3Xo] = VPNP_2_Krit3min(Xo(1:2))
[f4Xo] = VPNP_2_Krit4min(Xo(1:2))
disp('Оценка критериев в точке оптимума Xo в относительных единицах')%
L1Xo=(f1Xo+f1max)/d1
L2Xo=(f2Xo+f2max)/d2
L3Xo=(f3Xo+f3max)/d3
L4Xo=(f4Xo+f4max)/d4
% ****Конец*********
%    [Программа "Расчет 1 критер. - max"] файл: VPNP_2_Krit1max
 function [f,G] = VPNP_2_Krit1max(x)
f=-(x(1)-80).^2-(x(2)-80).^2;     %Расчет функции - критерий 1
G=[-2*(x(1)-80),    -2*(x(2)-80)];%Расчет 1 производной критерия 1
%  [Программа "Расчет 1 критер. - min"] Файл: VPNP_2_Krit1min
 function [f,G] = VPNP_2_Krit1min(x);
f=(x(1)-80).^2+(x(2)-80).^2;
G=[2*(x(1)-80);    2*(x(2)-80)];
% [Программа "Расчет 2 критер. - max"] Файл:  VPNP_2_Krit2max
 function [f,G] = VPNP_2_Krit2max(x);
f=-(x(1)-80).^2-(x(2)-20).^2;
G=[-2*(x(1)-80);    -2*(x(2)-20)];
%    [Программа "Расчет критер. 2 - min"] Файл: VPNP_2_Krit2min
 function [f,G] =VPNP_2_Krit2min(x);
f=(x(1)-80).^2+(x(2)-20).^2;
G=[2*(x(1)-80);    2*(x(2)-20)];
% [Программа "Расчет 3 критер. - max"] Файл: VPNP_2_Krit3max
 function [f,G] = VPNP_2_Krit3max(x);
f=-(x(1)-20).^2-(x(2)-20).^2;
G=[-2*(x(1)-20);    -2*(x(2)-20)];
% [Программа "Расчет 3 критер. - min"] Файл: VPNP_2_Krit3min
 function [f,G] = VPNP_2_Krit3min(x);
f=(x(1)-20).^2+(x(2)-20).^2;
G=[2*(x(1)-20);     2*(x(2)-20)];
% [Программа "Расчет 4 критер. - max"] Файл:VPNP_2_Krit4max
function [f,G] = VPNP_2_Krit4max(x);
f=-(x(1)-20).^2-(x(2)-80).^2;
G=[-2*(x(1)-20);    -2*(x(2)-80)];
```

```
% [Программа "Расчет 4 критер. - max"] Файл:VPNP_2_Krit4max
function [f,G] = VPNP_2_Krit4max(x);
f=-(x(1)-20).^2-(x(2)-80).^2;
G=[-2*(x(1)-20);    -2*(x(2)-80)];
%[Программа "Расчет критер. L-задачи"] файл: VPNP_2_L
function [f,G] = VPNP_2_L(x)
f=-x(3);
G=[0;     0;  -1];
%         [Программа "Расчет ограничений L-задачи"] файл: VPNP_1_LConst
function [c,ceq,DC,DCeq]= VPNP_2_LConst(x)
d1=12800;d2=12800;d3=12800;d4=12800;
f1X1max=12800;f2X2max=12800;f3X3max=12800;f4X4max=12800;
c(1)=((x(1)-80).^2+(x(2)-80).^2)/d1+x(3)-f1X1max/d1;
c(2)=((x(1)-80).^2+(x(2)-20).^2)/d2+x(3)-f2X2max/d2;
c(3)=((x(1)-20).^2+(x(2)-20).^2)/d3+x(3)-f3X3max/d3;
c(4)=((x(1)-20).^2+(x(2)-80).^2)/d4+x(3)-f4X4max/d4;
G1=[2*(x(1)-80)/d1, 2*(x(1)-80)/d2, 2*(x(1)-20)/d3, 2*(x(1)-20)/d4;
    2*(x(2)-80)/d1, 2*(x(2)-20)/d2, 2*(x(2)-20)/d3, 2*(x(2)-80)/d4;
             1.0,              1.0,           1.0,         1.0];
ceq=[]; DCeq=[];  % *****Конец*****
```

% *****Конец*****

Mathematical Preparation for Solving a Vector Problem of Nonlinear Programming

We will present the software for the VPNP solution on the example of solving a test vector problem of nonlinear programming.

***Example* 1.**

What is given. The consideration of the vector nonlinear (convex) programming problem with four homogeneous criteria. In terms of criteria, we use a circle, and with the linear restrictions the problem is therefore solved orally and imposed on variables.

$$opt\ F(X) = \{min\ F_2(X) = min\ f_1(X) \equiv (x_1 - 80)^2 + (x_2 - 80)^2, \tag{55}$$

$$min\ f_2(X) \equiv (x_1 - 80)^2 + (x_2 - 20)^2, \tag{56}$$

$$min\ f_3(X) \equiv (x_1 - 20)^2 + (x_2 - 20)^2, \tag{57}$$

$$min\ f_4(X) \equiv (x_1 - 20)^2 + (x_2 - 80)^2, \tag{58}$$

$$\text{at restriction } 0 \le x_1 \le 100,\ 0 \le x_2 \le 100, \tag{59}$$

What is required: To find the non-negative solution of x_1, x_2 in the system of inequalities (59), at which the $f_1(X), f_2(X)$ $f_3(X)$ and $f_4(X)$ functions perhaps accept the minimal value.

To solve the problem (55)-(59) for each criterion, as well as the further λ-problem, the MATLAB system (the function fmincon (…) – the solution to a non-linear problem of optimization) is used [43]. The solution is presented as a sequence of steps.

Step 1. VPLP (55)-(59) for each criterion is solved (max).

As a result of the decision we will receive:

Criterion 1: $X_1^* = \{x_1 = 0, x_2 = 0\}, f_1^* = f_1(X_1^*) = -12800;$

Criterion 2: $X_2^* = \{x_1 = 0, x_2 = 100\}, f_2^* = f_2(X_2^*) = -12800;$

Criterion 3: $X_3^* = \{x_1 = 100, x_2 = 100\}, f_3^* = f_3(X_3^*) = -12800;$

Criterion 4: $X_4^* = \{x_1 = 100, x_2 = 0\}, f_4^* = f_4(X_4^*) = -12800$**.**

Step 1. VPLP (55)-(59) for each criterion is solved (min).

As a result of the decision we will receive:

Step 2. VPLP (55)-(59) for each criterion is solved (min - opt).

As a result of the decision we will receive:

Criterion 1: $X_1^0 = \{x_1 = 80, x_2 = 80\}, f_1^0 = f_1(X_1^0) = 0;$

Criterion 2: $X_2^0 = \{x_1 = 80, x_2 = 20\}, f_2^0 = f_2(X_2^0) = 0;$

Criterion 3: $X_3^0 = \{x_1 = 20, x_2 = 20\}, f_3^0 = f_3(X_3^0) = 0:;$

Criterion 4: $X_4^0 = \{x_1 = 20, x_2 = 80\}, f_4^0 = f_4(X_4^0) = 0$**.**

Let's present a geometric interpretation of the limitations of the VPNP (55)-(59) and the results of the solution in Figure 1.

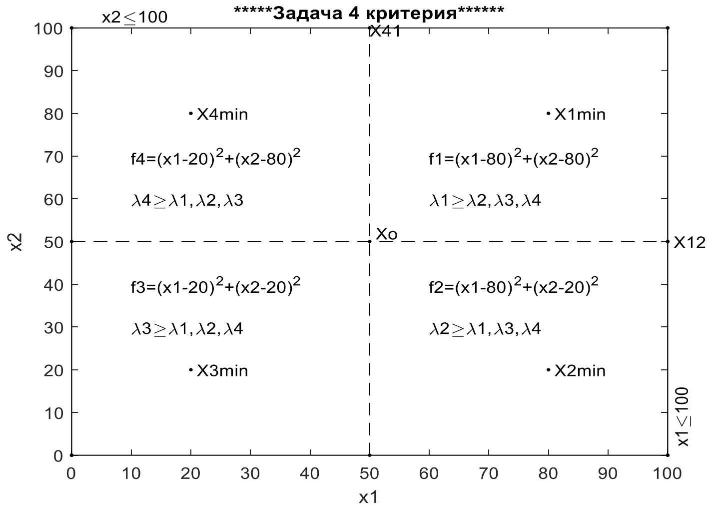

Figure 1. A geometrical interpretation of VPMP (55)-(59).
The results of the solution to VPMP

Step 3. A systems analysis of VPLP (55)-(59) criteria is made. For this purpose, at optimum points of $X_1^0 = X1min, X_2^0 = X2min, X_3^0 = X3min$ and $X_4^0 = 4min$ the values of objective functions $F(X^*)$ and the relative estimates are defined: $\lambda(X^*)$,

$\lambda_k(X) = \frac{f_k(X)-f_k^0}{f_k^*-f_k^0}, \forall k \in \boldsymbol{K}$:

$$F(X^*) = \{f_k(X_k^*), q = \overline{1,K}, k = \overline{1,K}, K = 4\} =$$

$$= \begin{vmatrix} f_1(X_1^*) & f_2(X_1^*) & f_3(X_1^*) & f_4(X_1^*) \\ f_1(X_2^*) & f_2(X_2^*) & f_3(X_2^*) & f_4(X_2^*) \\ f_1(X_3^*) & f_2(X_3^*) & f_3(X_3^*) & f_4(X_3^*) \\ f_1(X_4^*) & f_2(X_4^*) & f_3(X_4^*) & f_4(X_4^*) \end{vmatrix} = \begin{vmatrix} 0 & 3600 & 7200 & 3600 \\ 3600 & 0 & 3600 & 7200 \\ 7200 & 3600 & 0 & 3600 \\ 3600 & 7200 & 3600 & 0 \end{vmatrix},$$

$$\lambda(X^*) = \begin{vmatrix} \lambda_1(X_1^*) & \lambda_2(X_1^*) & \lambda_3(X_1^*) & \lambda_4(X_1^*) \\ \lambda_1(X_2^*) & \lambda_2(X_2^*) & \lambda_3(X_2^*) & \lambda_4(X_2^*) \\ \lambda_1(X_3^*) & \lambda_2(X_3^*) & \lambda_3(X_3^*) & \lambda_4(X_4^*) \\ \lambda_1(X_4^*) & \lambda_2(X_4^*) & \lambda_3(X_4^*) & \lambda_4(X_4^*) \end{vmatrix} = \begin{vmatrix} 1.0 & 0.7188 & 0.4375 & 0.7188 \\ 0.7188 & 1.0 & 0.7188 & 0.4375 \\ 0.4375 & 0.7188 & 1.0 & 0.7188 \\ 0.7188 & 0.4375 & 0.7188 & 1.0 \end{vmatrix}.$$

The Pareto set lies between optimum points $X_1^0 = X1min, X_2^0 = X2min, X_3^0 = X3min$ and $X_4^0 = 4min$, i.e., the area of admissible points of $\boldsymbol{S}$ formed by restrictions (59) coincides with a point set, which is Pareto-optimal $\boldsymbol{S^o}$, $\boldsymbol{S^o}=\boldsymbol{S}$.

At points of an optimum of $X_k^*, k = \overline{1,K}$ all relative estimates (the normalized criteria) are equal to zero: $\lambda_k(X_k^*) = \frac{f_k(X_k^*)-f_k^0}{f_k^*-f_k^0} = 0, k = \overline{1,K}, K = 4$.

At points of an optimum of $X_k^0, k = \overline{1,K}$ (anti optimum) all relative estimates are equal to the unit: $\lambda_k(X_k^0) = \frac{f_k(X_k^0)-f_k^0}{f_k^*-f_k^0} = 1, k = \overline{1,K}, K = 4$. From here, $\forall k \in \boldsymbol{K}, \forall X \in \boldsymbol{S}, 0 \le \lambda_k(X) \le 1$.

Step 4. λ-problem is under construction:

$$\lambda^o = \boldsymbol{max}_{X \in S}\, \lambda, \tag{60}$$

at restrictions: $\lambda - \frac{f_1(X)-f_1^0}{f_1^*-f_1^0} \le 0, \lambda - \frac{f_2(X)-f_2^0}{f_2^*-f_2^0} \le 0, \lambda - \frac{f_3(X)-f_3^0}{f_3^*-f_3^0} \le 0, \lambda - \frac{f_4(X)-f_4^0}{f_4^*-f_4^0} \le 0,$

$0 \le x_1 \le 100,\ 0 \le x_2 \le 100.$

Step 5. *Solution to the λ-problem*. The solution to the λ-problem in the MATLAB system is similar to step 0 initial parameters

The appeal to the fmincon (…) function for the solution to the λ-problem is presented. Results of the solution to the λ-problem:

$\boldsymbol{X^o} = \{\, x_1 = 50.0, x_2 = 50.0, x_3 = 0.8594\}$**.**

where $\boldsymbol{X^o}$ defines the best values of variables; coordinate of x_1 corresponds to λ^o– maximum relative level: $x_3 = \lambda^o$; and x_1, x_2 corresponds x_1, x_2 to of a problem ((41)-(45).

Lo $= \lambda^o =$ 0.8594 represents the optimal value of the objective function.

λ^o is the maximum level among all minimum relative levels on an admissible set of $X \in \boldsymbol{S}$:

$\lambda^o = \boldsymbol{max}_{\boldsymbol{X\in S}} \boldsymbol{min}_{\boldsymbol{k\in K}} \lambda_k(X).$

The functions $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$, and points of an optimum of X^o and λ^o which are received on their crossing are shown in Figure 2.

As in a task (55)-(59) criteria are symmetric and the point of the guaranteed result of $X^o = \{x_1 = 50., x_2 = 50.\}$ is easily defined – it lies in the center of a square, and a maximal relative assessment of $\lambda^o = 0.5833$. Really, for example, with the first criterion:

$$\lambda_1(X^o) = \frac{f_1(X^o) - f_1^0}{f_1^* - f_1^0} = ((50-80)^2 + (50-80)^2 - 12800)/(0 - 12800) = 0.8594,$$

and similarly for other criteria:

$\lambda^o = \lambda_1(X^o) = \lambda_2(X^o) = \lambda_3(X^o) = \lambda_4(X^o) = 0.8594.$

This is very clearly shown in Figure 4.

In Figures 3 and 4 we can see that the area (point set) limited to function:

$f1 = (x_1 - 80)^2 + (x_2 - 80)^2$ is characterized by $\lambda_1(X) \ge \lambda_k(X), k = \overline{2,4}$, $X \in \boldsymbol{S_1}$, (Figure 3 shows how λ1>λ2, λ3, λ4), i.e., it is prioritized by the first criterion. In this area, the priority of the first criterion is always more or equal to the unit: $p_k^1(X) = \lambda_1(X)/\lambda_k(X) \ge 1,\ \forall X \in \boldsymbol{S_1}$.

Areas (point set) prioritized by the corresponding criterion are similarly shown. In total, they give a point set, which is Pareto-optimal, of $\boldsymbol{S^o}$, and it (for this example) is equal to a set of admissible points: $\boldsymbol{S^o} = \boldsymbol{S_1^o} \cup \boldsymbol{S_2^o} \cup \boldsymbol{S_3^o} \cup \boldsymbol{S_4^o} \cup \boldsymbol{X^o} = \boldsymbol{S}$.

2.4.3. ***Study of the Results of Solving a Vector Problem of Nonlinear Programming as a Mathematical Model of an Engineering System***

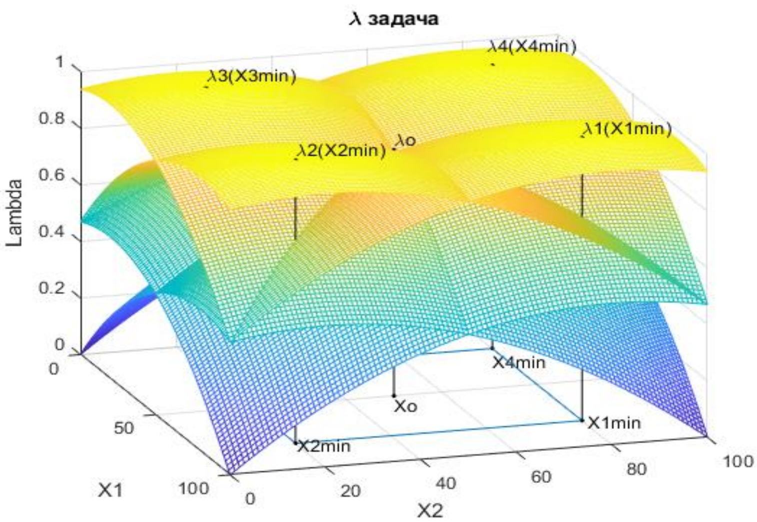

Figure 2. Results of the solution to VPLP (55)-(59) and (60): functions: $\lambda_1(X), \lambda_2(X), \lambda_3(X), and\ \lambda_4(X)$, points of an optimum of X^o and λ^o

By the concept of an engineering system, we understand: technical systems, technological processes, material structure. A vector problem of nonlinear (convex) programming with four homogeneous criteria (55)-(59) and the corresponding λ-problem (60) are considered as a mathematical model of an engineering system. Functions in relative units $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$ as well as the optimum point X^o and λ^o, which are obtained at the intersection, in the three-dimensional coordinate system ,x-1., ,x-2.,λ are shown in Fig. 2.

Let us imagine a solution with equivalent criteria in relative units - λ-problems. As a result of the solution with equivalent criteria (-the problems received the optimum point:

$\boldsymbol{X^o} = \{ x_1 = 50.0, x_2 = 50.0, x_3 = 0.8594\}$, where x_3=λ°= 0.8594.

λ° =0.8594 is a guaranteed result in relative units, which shows that an improvement in any criterion leads to a deterioration in other criteria.

Relative units are easily converted into physical units for each criterion (characteristic of the engineering system). The third step is to solve the VPMP with equivalent criteria at the optimum points X_1^*, X_2^*, X_3^*, X_4^*. (in Fig. 2: $X1min, X2min, X3min, X4min$) the values of all relative estimates are obtained:

$$L(Y^*) = \left\| \begin{matrix} L(X_1^*) \\ L(X_2^*) \\ L(X_3^*) \\ L(X_4^*) \end{matrix} \right\| = \left\| \begin{matrix} \mathbf{1.0}\ 0.7188\ 0.4375\ 0.7188 \\ 0.7188\ \mathbf{1.0}\ 0.7188\ 0.4375 \\ 0.4375\ 0.7188\ \mathbf{1.0}\ 0.7188 \\ 0.7188\ 0.4375\ 0.7188\ \mathbf{1.0} \end{matrix} \right\|.$$

Let's imagine from Fig. 2 values of relative estimates at the optimum point, for example X_2^*: $L(X_2^*) = \{\lambda_1(X_2^*) = 0.7188, \lambda_2(X_2^*) = \mathbf{1.0}, \lambda_3(X_2^*) = 0.7188, \lambda_4(X_2^*) = 0.4375\}$.

A linear function connecting the points λ^o=0.8594 and $\lambda_2(X_2^*)$=1 in relative units characterizes the function $f_2(X)$ in relative units in the two-dimensional dimension of the parameters x_1, x_2. This linear function represents a geometric interpolation of the functions $f_2(X)$ in relative units from N-dimensional (in this example, 2-dimensional) to a two-dimensional coordinate system.

Suppose that for the designer, the function $f_2(X)$ is the most important-priority. In the study of the priority function $f_2(X)$ we can set different numeric values to $f_2(X)$ from the ratio: $f_2(\boldsymbol{X^o}) \leq f_2(X) \leq f_2(X_2^*)$ and as a result we can obtain the corresponding parameters.

In the same way, you can study other criteria (functions of an engineering system): $f_1(X), f_3(X), f_4(X)$.. The numerical solution of such a study is presented on a four-dimensional system (material structure).

Modeling, simulation and optimal decision making based on a numerical set of criteria: a technological process with two parameters

3. Research, modeling and selection of optimal parameters of the technological process in conditions of certainty and uncertainty on the basis of multidimensional mathematics.

The numerical implementation of the choice of optimal parameters of the technological process is carried out on the basis of the theory of multidimensional mathematics methods. The methodology of numerical implementation of the selection of optimal parameters of the technological process is presented as a number of stages, which are divided into separate blocks, steps.

3.1. Mathematical model of the technological process under conditions of certainty and uncertainty in the aggregate

The mathematical model of the technological process is presented in accordance with the formed mathematical model under the conditions of certainty (1b)-(4b) and uncertainty (5b)-(10b) in the form of a vector problem of mathematical programming:

$$Opt\ F(X) = \{max\ F_1(X) = \{max\ f_k\ (X), k = \overline{1, K_1^{def}}\}, \quad (61)$$

$$max\ I_1(X) \equiv \{max\ y_k\ (X_i,\ i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}, \quad (62)$$

$$min\ F_2(X) = \{min\ f_k\ (X), k = \overline{1, K_2^{def}}, \quad (63)$$

$$min\, I_2(X) \equiv \{min\, y_k\,(X_i,\ i = \overline{1,M})\}^T, k = \overline{1, K_1^{unc}}\}, \quad (64)$$

$$\text{при ограничениях} \quad f_k^{min} \le f_k(X) \le f_k^{max}, k = \overline{1,K}, \quad (65)$$

$$G(X) \le 0,\ x_j^{min} \le x_j \le x_j^{max}, j = \overline{1,N}, \quad (66)$$

where $X = \{x_j,\ j = \overline{1,N}\}$ is the vector of controlled variables (parameters) of the technological process; $F(X) = \{F_1(X)\, F_2(X)\, I_1(X)\, I_2(X)\}$ represents a vector criterion, each component of which is a vector of characteristics (criteria) of the technological process (6b)-(10b); The criteria $F_1(X)\, F_2(X)$ functionally depend on the vector values of the variables X, which are denoted as (61) and (63) K_1^{def}, K_2^{def} (*definiteness*). Formed in conditions of certainty;

The criteria $I_1(X)\, I_2(X)$ characterize discrete values of functions that are functionally dependent on discrete values of the vector of variables X, denoted in (62) and (64) as K_1^{unc}, K_2^{unc} (*uncertainty*) set of max and min criteria. Formed in conditions of uncertainty. (65) and (66) standard constraints: functional, parametric and general.

3.2. Terms of reference for the formation of a mathematical and, on its basis, numerical model of the technological process.

The terms of reference, the analysis of the results of the solution and the choice of the priority criterion, its magnitude are carried out by the material *designer*. The remaining stages are performed by *a mathematician-programmer*.

3.2.1. The technical assignment: The choice of the optimal parameters of the technological process

It is given. The technological process, the operation of which is determined by two parameters $X = \{x_1, x_2\}$ - the vector of variables (controlled): 2.0 ≤ x_1 ≤ 3.5, 12.0 ≤ x_2≤ 30.0.

The operation of the process is determined by four characteristics (criteria):

$F(X) = \{f_1(X), f_2(X), f_3(X), f_4(X)\}^T$, the value of which depends on the vector of parameters X.

Conditions of certainty. The conditions are characterized by the fact that the functional dependence of the fourth characteristic $f_4(X)$ on the parameters of the technological process $X = \{x_1, x_2\}$ is known:

$$f_4(X) = -0.2450 - 0.7470x_1 + 0.3832x_1^2 + 0.0442x_2 + 0.0012x_2^2 - 0.0346x_1x_2. \quad (67)$$

Conditions of uncertainty. For the first, second and third characteristics of the technological process, the results of experimental data are known: the values of the parameters and the corresponding characteristics. The numerical values of the parameters $X = \{x_1, x_2\}$ and the characteristics $f_1(X), f_2(X), f_3(X)$ are presented in Table 2.

Table 2.

Experimental weld input and output parameters.

Laser Power, p (Analog V)	Travel Speed, v (mm/sec)	Wire Feed Rate, r (m/min)	Depth, D(mm)	Total Accumulated Pore Length, Po (mm/mm)
x_1	x_2	$y_1(X) \to max$	$y_2(X) \to min$	$y_3(X) \to max$
2.40	24.2	4.2197	55.3951	-0.0365
2.76	18.72	3.2714	31.2497	0.0286
2.76	19.08	3.2770	32.3886	0.0271
2.76	31.68	4.2613	86.8526	0.0760
2.76	31.92	4.2949	88.1656	0.0787
3.30	14.40	3.0959	21.1331	0.3467
3.30	25.20	3.0101	56.1913	0.2171
3.30	25.80	3.0383	58.7506	0.2138
3.30	26.76	3.0908	62.9794	0.2096
3.30	27.60	3.1441	66.8147	0.2068

3.30	28.80	3.2320	72.5126	0.2041
3.30	30.00	3.3338	78.4682	0.2032
3.30	31.20	3.4495	84.6812	0.2039
3.30	32.40	3.5792	91.1518	0.2063
3.30	36.00	4.0517	112.1086	0.2236
3.84	18.72	3.0983	35.3082	0.6402
3.84	23.52	2.9671	51.6458	0.5450
3.84	31.68	3.2554	88.8758	0.4451
3.84	32.88	3.3520	95.3551	0.4369
4.20	25.20	3.2370	60.4633	0.7810
Max: 2.40	14.4	2.9671	21.1331	-0.0365
Max: 4.20	36.0	4.2949	112.1086	0.7810

In the decision taken, the evaluation value for the first, second and third characteristics (criteria) are desirable to get as high as possible: $y_1(X) \to \mathbf{max}, y_2(X) \to \mathbf{min}, y_3(X) \to \mathbf{max}, f_4(X) \to \mathbf{min}$. The parameters $X = \{x_1, x_2\}$ vary within the following limits:

$$2 \le x_1 \le 3.5, 12 \le x_2 \le 30. \quad (68)$$

It is required. Build a model of the technological process in the form of a vector problem. Solve the vector problem with equivalent criteria. Choose a priority criterion. Set the numerical value of the priority criterion. Solve the vector optimization problem and make the best (optimal) solution with the given criterion priority.

3.2.2. Construction of a mathematical model of a technological process in conditions of certainty and uncertainty in a general form

The mathematical model of the technological process under conditions of certainty and uncertainty is developed in the form of a vector problem of mathematical programming and the general form of the model is presented in section 3.1.

3.3. Construction of a numerical model of a technological process with functional dependence on parameters (conditions of certainty and uncertainty).

The development of a mathematical "Model of the technological process under conditions of certainty and uncertainty" in the form of a vector optimization problem (5b)-(10b) is presented in Section 1.2.

3.3.1. Construction of a mathematical (numerical) model of a technological process under conditions of certainty

The construction of a model of a technological process with functional dependence is determined by the fact that the characteristics and limitations of the parameters $X = \{x_1, x_2\}$ are known. Characteristics (67) and limitations (68) are known. Using data (67), (68), we construct a one-criteria problem of nonlinear programming - conditions of certainty:

$$f_4(X) = -0.245 - 0.747x_1 + 0.3832x_1^2 + 0.0442x_2 + 0.0012x_2^2 - 0.0346x_1 * x_2, \quad (69)$$

$$2.0 \le x_1 \le 3.5, 12.0 \le x_2 \le 30.0. \quad (70)$$

3.3.2. Transformation of uncertainty conditions (experimental data) into conditions of certainty and construction of a numerical model

Digital transformation consists in the use of qualitative and quantitative descriptions of the technological system obtained on the principle of "input-output" in Table 2. Converting information (initial data $f_1(X), f_2(X), f_3(X)$) into a functional form $f_1(X)$ $f_2(X)$ $f_3(X)$ is carried out by using mathematical methods (the regression analysis). The initial data of table 2 are formed as a matrix *I* in the MATLAB system:

$$I = |X,F| = \begin{vmatrix} X_1 = \{x_{11}, x_{12}\}, \; f_1(X_1), f_2(X_1), f_3(X_1) \\ \dots \\ X_M = \{x_{M1}, x_{M2}\}, f_1(X_M), f_2(X_M), f_3(X_M) \end{vmatrix}. \quad (71)$$

For each experimental data set $f_k, k = \overline{1,3}$ the regression function is constructed using the least squares method $\mathbf{min} \sum_{i=1}^{M} (f_i - \bar{f_i})^2$ in the MATLAB system. For this, the polynomial A_k is formed, which determines the interrelation of the parameters $X_i = \{x_{i1}, x_{i2}\}$ and the function

$$\overline{f_{ki}} = f(X_i, A_k), X_{ki} = \{x_{ki}, x_{ki}\}, k = \overline{1,3}. \quad (72)$$

The result is a system of coefficients: $A_k = \{A_{0k}, A_{1k}, \dots, A_{9k}\}$,

which determine the coefficients of the polynomial (function):

$$f_k(X, A) = a_{0k} + a_{1k}x_1 + a_{2k}x_1^2 + a_{3k}x_2 + a_{4k}x_2^2 + a_{5k}x_1 * x_2, \; k = \overline{1,3}. \quad (73)$$

To determine the polynomial coefficients (73) of the functions in Table 2, polynomial approximation software with two variables and five factors is developed.

As a result of the program's work, the coefficients were obtained:

```
Ao=[11.4751   8.8173  -0.1222      % A0k
   -4.8994  -7.6807  -0.3735      % A1k
    0.8868   2.1456   0.1916      % A2k
   -0.0030   0.1851   0.0221      % A3k
    0.0048   0.0894   0.0006      % A4k
   -0.0595  -0.1454  -0.0173].    % A5k
```
(74)

As a result, $\{x_{1i}, x_{2i}, f_{1i}, i = \overline{1,M}\}$ - the experimental data of the matrix $I = |X,F|$ (74) are transformed into the function $f_1(X)$ in the form (73), which, taking into account the obtained coefficients, will take the form:

$$f_1(X) = 11.474 - 4.8992x_1 + 0.8868x_1^2 - 0.0030\,x_2 + 0.0048x_2^2 - 0.0595x_1x_2.$$

The experimental data $I_2 = \{x_{i1}, x_{i2}, f_{i2}\}$ of the matrix $I = |X,Y|$ (74) are converted into the function $f_2(X)$ in the form (73), which, taking into account the obtained coefficients, takes the form:

$$f_2(X) = 8.817 - 7.6809x_1 + 2.1456x_1^2 + 0.1851x_2 + 0.0894x_2^2 - 0.1454\,x_1x_2.$$

The experimental data $I_3 = \{x_{i1}, x_{i2}, f_{i3}\}$ of the matrix $I = |X,F|$ (74) are converted to the function $f_3(X)$ in the form (73), which, taking into account the obtained coefficients, takes the form:

$$f_3(X) = -0.1225 - 0.3735x_1 + 0.1916x_1^2 + 0.0221x_2 + 0.0006x_2^2 - 0.0173x_1x_2.$$

As a result, the experimental data of Table 1 are formed taking into account the purposefulness of the vector problem of mathematical programming:

$$max\; f_1(X) = 11.474 - 4.899x_1 + 0.8868x_1^2 - 0.003x_2 + 0.0048x_2^2 - 0.0595x_1x_2. \quad (75)$$

$$max\; f_2(X) = 8.817 - 7.681x_1 + 2.145x_1^2 + 0.1851x_2 + 0.0894x_2^2 - 0.1454x_1x_2. \quad (76)$$

$$max\; f_3(X) = -0.12251 - 0.3736x_1 + 0.1916x_1^2 + 0.0222x_2 + 0.0006x_2^2 - 0.0173x_1x_2.$$

$$2.0 \le x_1 \le 3.5,\; 12.0 \le x_2 \le 30.0. \quad (77)$$

3.3.3. Mathematical numerical model of the technological process with aggregated experimental data under conditions of certainty

(General part for the conditions of certainty and uncertainty).

To build a mathematical model of a technological process, we use: the functions obtained by the conditions of certainty and uncertainty (69), (75)-(77); parametrical restrictions (77).

Functions (69), (75)-(77) are considered as criteria determining the purposefulness of the operation of the technological process. All criteria are aimed $\boldsymbol{K}_1 = 2$ at maximization: $f_1(X), f_3(X)$→max and $\boldsymbol{K}_2$ = 2 at minimization: $f_2(X), f_4(X)$→min $\boldsymbol{K} = \boldsymbol{K}_1 \cup \boldsymbol{K}_2$. As a result, the model of the functioning of the technological process will be represented by the vector problem of mathematical programming:

$$Opt\; F(X) = \{max\; F_1(X) = \{max\; f_1\,(X) \equiv 11.474 - 4.899x_1 + 0.8867x_1^2$$
$$- 0.0031x_2 + 0.0048x_2^2 - 0.0595x_1x_2, \quad (78)$$

$$max\ f_3\ (X) \equiv -0.1225 - 0.3735x_1 + 0.1916x_1^2 + 0.0221x_2 + 0.0006x_2^2 - 0.0173x_1x_2, \quad (79)$$

$$min\ F_2(X) = \{min\ f_2\ (X) \equiv 8.817 - 7.68x_1 + 2.1456x_1^2 + 0.185x_2 + 0.0894x_2^2 - 0.145x_1x_2, \quad (80)$$

$$min\ f_4(X) \equiv -0.2451 - 0.7471x_1 + 0.3832x_1^2 + 0.0442x_2 + 0.0012x_2^2 - 0.0346x_1x_2\}\}, \quad (81)$$

restrictions: $2.01 \le x_1 \le 3.5,\ 12.01 \le x_2 \le 30.01,$ (82)

The vector problem of mathematical programming (78)-(82) represents a mathematical model of making optimal decisions in conditions of certainty and uncertainty in the aggregate, as a result of digital transformation.

3.4. Optimal Decision Making in a Two-Parameter Four-Dimensional Process Model.

3.4.1. Mathematical Modeming of the Technological Process with Equivalent Criteria.

To solve the vector problem of mathematical programming (78)-(82), we use the solution method based on the normalization of criteria and the principle of guaranteed results, presented in section 4. The solution of the vector problem (78)-(82) with equivalent criteria is represented as a sequence of steps.

Step 1. Solving VPMP (78)-(82) for each criterion separately, using the fmincon(...) function of the MATLAB system [19], the call to the fmincon(...) function is considered in [15]. As a result of the calculation for each criterion, we obtain optimum points: X_k^* and $f_k^* = f_k(X_k^*), k = \overline{1, K}, \boldsymbol{K} = 4$ is the values of the criteria at this point, that is, the best solution for each criterion:

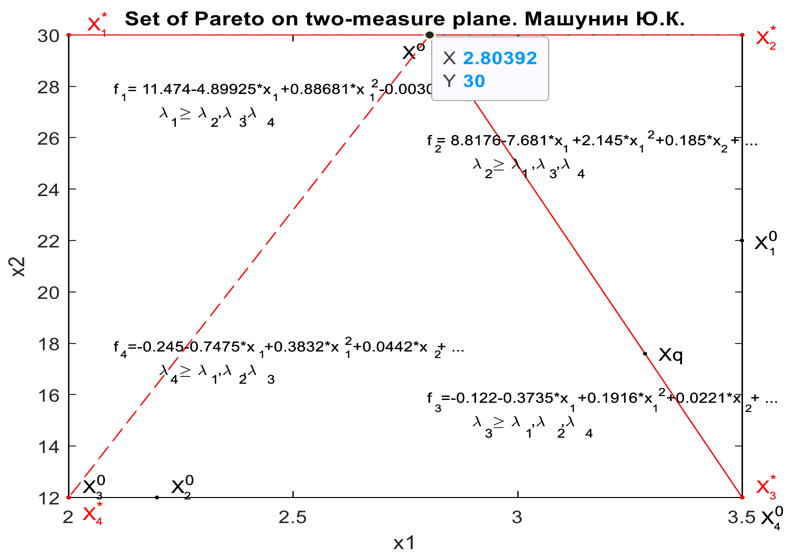

Figure 3. Pareto set, $S^o \subset S$ in two-dimensional coordinate system, the points of the optimum $X_1^*, X_2^*, X_3^*, X_4^*$.

$$X_1^* = \{x_1 = 2, x_2 = 30\}, f_1^* = f_1(X_1^*) = -5.8833;$$
$$X_2^* = \{x_1 = 3.5, x_2 = 30\}, f_2^* = f_2(X_2^*) = -78.9641;$$
$$X_3^* = \{x_1 = 3.5, x_2 = 12.0\}, f_3^* = f_3(X_3^*) = -0.54235;$$
$$X_4^* = \{x_1 = 2.9, x_2 = 12.0\}, f_4^* = f_4(X_4^*) = -0.3334. \quad (83)$$

The constraints (82) and optimum points $X_1^*, X_2^*, X_3^*, X_4^*$ (83) in the coordinates $\{x_1, x_2\}$ are shown in Figure 11.1. The set of valid points $\boldsymbol{S}$ is not empty and is a compact: $\boldsymbol{S} = \{X \in \boldsymbol{R}^N \mid 2.0 \le x_1 \le 3.5, 12.0 \le x_2 \le 30\} \neq \emptyset$.

The set of points that are Pareto optimal, $\boldsymbol{S^o}$ represents the area of the set of points that lie between the points of the optimum $X_1^*, X_2^*, X_3^*, X_4^*$. We see that in this problem, the set of admissible points $\boldsymbol{S}$ and the set of points optimal in Pareto, $\boldsymbol{S^o}$, are equal to each other: $\boldsymbol{S} = \boldsymbol{S^o}$.

Step 2. The worst unchanging part of each criterion (anti-optimum) is determined (a superscript zero): $X_1^0 = \{x_1 = 3.50, x_2 = 22.0052\}, f_1^0 = f_1(X_1^0) = 2.8663;$

$$X_2^0 = \{x_1 = 2.1952, x_2 = 12.0\}, f_2^0 = f_2(X_2^0) = 13.56;$$

$X_3^0 = \{x_1 = 2.0, x_2 = 12.0\}, f_3^0 = f_3(X_3^0) = -0.1667;$
$X_4^0 = \{x_1 = 3.5, x_2 = 12.0\}, f_4^0 = f_4(X_4^0) = -1.0847.$ (84)

Step 3. A systematic analysis of the set of Pareto optimal points (i.e. analysis by each criterion) is performed.

At the optimum points $X^* = \{X_1^*, X_2^*, X_3^*, X_4^*\}$, the values of the objective functions $F(X^*) = \|f_q(X_k^*)\|_{q=\overline{1,K}}^{k=\overline{1,K}}$ are determined, the vector $D = (d_1\ \ d_2\ d_3\ d_4)^T$deviations for each criterion on the admissible set $\boldsymbol{S}$: $d_k = f_k^* - f_k^0, k = \overline{1,4}$, $d_k = \{3.017, 65.4, 0.709, 1.418\}$ $\lambda(X^*) = \|\lambda_q(X_k^*)\|_{q=\overline{1,K}}^{k=\overline{1,K}}$, where $\lambda_k(X) = (f_k^* - f_k^0)/d_k$.

$$F(X^*) = \left\|\begin{matrix} -5.883 - 79.327 - 0.0619\ 0.123 \\ -3.173 - 78.964 - 0.3038\ 0.607 \\ -3.346 - 17.206 - 0.5423\ 1.084 \\ -4.450 - 13.64 - 0.1667 - 0.333 \end{matrix}\right\|, \lambda(X^*) = \left\|\begin{matrix} 1.0\ 1.0056\ 0.3224\ 0.6776 \\ 0.1017\ 1.0\ 0.6636\ 0.3364 \\ 0.1593\ 0.0557\ 1.0\ 0.0 \\ 0.5251\ 0.0013\ 0.0\ 1.0 \end{matrix}\right\|, \quad (85)$$

where $X^* = \{X_1^*, X_2^*, X_3^*, X_4^*\}$.

The analysis of the values of criteria and relative estimates (85) shows that at the points of optimum X^*(diagonally) the relative estimate is equal to one. The remaining criteria are significantly less than one. It is required to find a point (parameters) at which the relative estimates are closest to unity. The solution of this problem is aimed at solving the λ-problem - step 4, 5.

Step 4. The construction of the λ-problem is carried out in two stages: a maximin optimization problem with normalized criteria is initially constructed:

$$\Lambda^o = \boldsymbol{max}_{X\in S}\boldsymbol{min}_{k\in K}\lambda_k(X), G(X)\le 0, X \ge 0. \quad (86)$$

At the second stage, the maximin problem (86) is transformed into a standard mathematical programming problem:

$$\lambda^o = max\ \lambda, \quad (87)$$

at restrictions

$$\lambda - (11.4745 - 4.8992x_1 + 0.8868x_1^2 - 0.003x_2 + 0.0048x_2^2 - 0.0595x_1x_2 - f_1^0)/\ d_1 \le 0, \quad (88)$$
$$\lambda - (-0.1225 - 0.3735x_1 + 0.1916x_1^2 + 0.022x_2 + 0.0006x_2^2 - 0.0173x_1x_2 - f_2^0)/\ d_2 \le 0, \quad (89)$$
$$\lambda - (8.8176 - 7.6809x_1 + 2.1456x_1^2 + 0.1851x_2 + 0.0894x_2^2 - 0.1454x_1x_2 - f_3^0)/\ d_3 \le 0, \quad (90)$$
$$\lambda - (-0.245 - 0.747x_1 + 0.3832x_1^2 + 0.0442x_2 + 0.0012x_2^2 - 0.0346x_1x_2 - f_4^0)/\ d_4 \le 0, \quad (91)$$

at resignment $2.01 \le x_1 \le 3.5$, $12.01 \le x_2 \le 30.01$, (92)

where the vector of unknowns has dimension of $N + 1$: $\boldsymbol{X} = \{x_1, \dots, x_N, \lambda\}, N = 2$.

Step 5. Solution of the λ-problem. For this purpose we use the function fmincon(…), [17]: $[\mathrm{Xo, Lo}] = \mathrm{fmincon}('\mathrm{Z\_TS\_2Krit\_L}',\ \mathrm{X0, Ao, bo, Aeq, beq, lbo, ubo}, '\mathrm{Z\_TS\_LConst}', \mathrm{options})$

The result of solving the vector optimization problem (78)-(82) with equivalent criteria, with the corresponding λ-problem (87)-(92) was obtained:

The optimum point X^o with equivalent criteria

$$\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 2.8039, x_2 = 30.0, \lambda^o = 0.3541\}\} \quad (93)$$

which represents the design parameters of the technological process with equivalent criteria (characteristics). The optimal point is X^o is shown in Figure 3;

The values of the criteria $f_k(X^o), k = \overline{1,K}$ at the optimum point X^o:

$$F(X^o) = \{f_1(X^o) = 3.9345, f_2(X^o) = 77.9319, f_3(X^o) = 0.0844, f_4(X^o) = 0.1687\}; \quad (94)$$

which represent the characteristics of the technological process;

Values of relative estimates $\lambda_k(X^o), k = \overline{1,K}$ at the optimum point X^o:

$$\lambda(X^o) = \{\lambda_1(X^o) = 0.3541, \lambda_2(X^o) = 0.9842,\ \lambda_3(X^o) = 0.3541, \lambda_4(X^o) = 0.6459\}; \quad (95)$$

Maximum lower level λ^o $=\ 0.3541$ at the optimum point X^o: which is measured in relative units, among all relative estimates:

$\lambda^o = min\,\{\lambda_1(X^o), \lambda_2(X^o), \lambda_3(X^o), \lambda_4(X^o)\} \; = \; 0.3541,$

λ^o - also called a guaranteed result in the relative units, i.e., $\lambda_k(X^o)\;\; \forall k \in \boldsymbol{K}$ and, accordingly, the characteristics of the technological process $f_k(X^o)\;\; \forall k \in \boldsymbol{K}$ it is impossible to improve, without worsening at the same time other characteristics.

In accordance with Theorem 1, at point $\boldsymbol{X^o}$ (95), criteria 1 and 3 are contradictory. This contradiction is determined by the equality: $\lambda_1(X^o), \lambda_3(X^o) = \lambda^o = 0.3541,$

and the remaining criteria by the inequality $\{\lambda_2(X^o) = \; 0.9842, \lambda_4(X^o) = \; 0.6459\} > \lambda^o.$

3.4.2. Creation of geometrical interpretation of results of the decision in a three-dimensional coordinate.

In the allowable set of points $\boldsymbol{S}$ formed by constraints (82), the optimum points $X^* = \{X_1^*, X_2^*, X_3^*, X_4^*\}$, which are shown in Fig. 3, combined into a contour, represent the set of Pareto optimal points, $\boldsymbol{S^o} \subset \boldsymbol{S}$. The coordinates of these points, as well as the characteristics of the technological process in relative units $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$ are shown in **Figure 4** in the three-dimensional space x_1 , x_2 and λ, where the third axis is the relative estimate λ.

3.4.3. Decision-making in the technological process model at the set priority of criterion

To solve vector problems of mathematical programming (78)-(82) methods are presented, based on axiomatic normalization of criteria and principle of guaranteed result, as well as axiomatic priority of criterion, resulting from axiom 2, 3 and principle of optimality 2, which are presented in section 5, [19]. The decision maker is usually the process designer.

Step 1. *We solve a vector problem* (78)-(82) *with equivalent criteria*. The algorithm of the decision is presented in the stage 3. Numerical results of the solution of the vector problem are given above. Pareto's great number of $S^o \subset$ S lies between optimum points $X_1^*, X_2^*, X_3^*, X_4^*, X_1^*$. This information is the basis for further research on the structure of the Pareto set. The decision maker is usually the technological process designer.

If results of the solution of a vector problem with equivalent criteria do not satisfy the person making the decision, then the choice of an optimal solution is carried out from any subset of points of $\boldsymbol{S_1^o}, \boldsymbol{S_2^o}, \boldsymbol{S_3^o}, \boldsymbol{S_4^o}$, **Figure 4**.

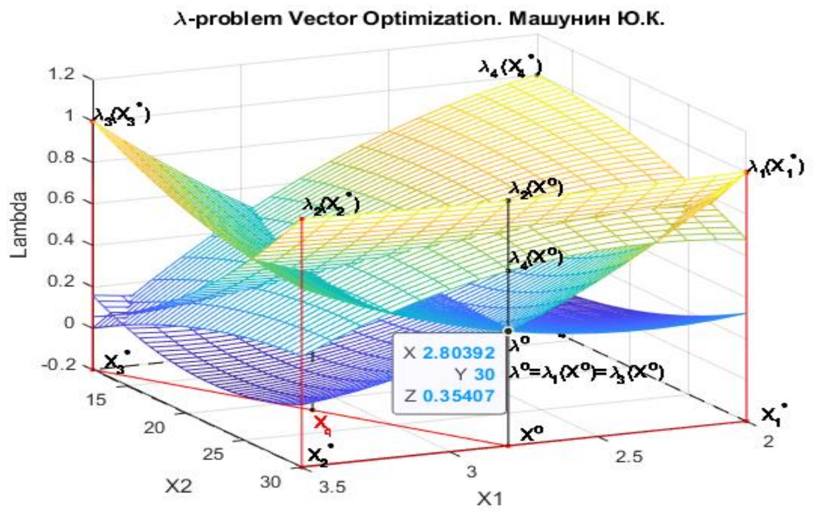

Figure 4. Results of solution λ-problems $\lambda_1(Y), \lambda_2(Y), \lambda_3(Y),$ $\lambda_4(Y)$ in the three-dimensional coordinate system x_1, x_2 and λ

Step 2. Choice of priority criterion of $q \in \boldsymbol{K}$. From the theory (the Theorem 2) it is known that in an optimum point of X^o there are always two most contradictory criteria: $q \in \boldsymbol{K}$ and $v \in \boldsymbol{K}$ for which in the relative unit's precise equality is carried out:

$\lambda^o = \lambda_q(X^o) = \lambda_p(X^o), q, p \in \boldsymbol{K}, X \in \boldsymbol{S}$,

and for the others it is carried out inequalities: $\lambda^o \le \lambda_k(X^o) \;\; \forall k \in \boldsymbol{K}$.

For the choice of the priority criterion on the display the message about results of the solution of λ-problem in physical and relative units is given: Criteria (93) in X^o optimum point:

$F(X^o) = \{f_1(X^o) = 3.9345, f_2(X^o) = 77.9319, f_3(X^o) = 0.0844, f_4(X^o) = 0.1687\}$;

The relative estimates in X^o:

$\lambda(X^o) = \{\lambda_1(X^o) = 0.3541, \lambda_2(X^o) = 0.9842, \lambda_3(X^o) = 0.3541, \lambda_4(X^o) = 0.6459\}$.

From the function $\lambda(X^o)$ it is clear that the first and the third are the most contradictory criteria:

$$\lambda^o = \{\lambda_1(X^o), \lambda_3(X^o)\} = 0.3541, \quad (96)$$

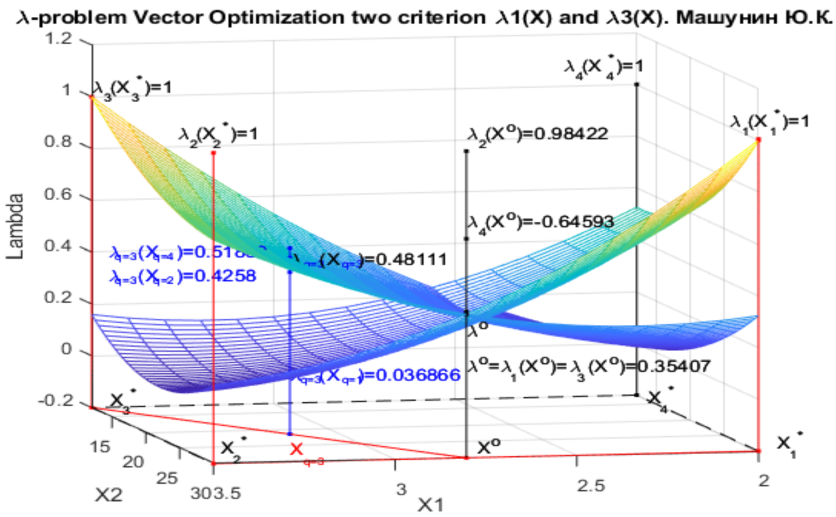

Figure 5. Result the solution of the problem (78)-(82) in the three-dimensional coordinate system $\{x_1, x_2, \lambda\}$. $\lambda_1(X^o) = \lambda_2(X^o) = \lambda_3(X^o)$=0. 3542.

Select from **Figure 4** the first and third criteria and present $\{\lambda_1(X^o), \lambda_3(X^o)\}$ in the relative units in **Figure 5**.

As a rule, from a pair of the contradictory criteria, a criterion chosen by the decision maker would be improved. Such criterion is called "priority criterion", we will designate it $q = 3 \in \boldsymbol{K}$. This criterion is investigated in interaction with the first criterion of $k = 1 \in \boldsymbol{K}$.

On the display the message is given:

q=input ('Enter priority criterion (number) of q =') - Entered: *q*=3.

Step 3. *Numerical limits* of the change of the size of a priority of the criterion of $q = 3 \in \boldsymbol{K}$ are defined.

For priority criterion of $q = 3 \in \boldsymbol{K}$ changes of the numerical limits in the physical units upon transition from X^o optimum point to the point of X_q^* received on the first step at equivalent criteria are defined. *q*=3 given about criterion are given for the screen:

$$f_q(X^o) = -0.084 \le f_q(X) \le -0.54 = f_q(X_q^*), q = 3 \in \boldsymbol{K}. \quad (97)$$

In the relative units the criterion of *q*=3 changes in the following limits:

$$\lambda_q(X^o) = 0.35407 \le \lambda_q(X) \le 1 = \lambda_q(X_q^*), q = 3 \in \boldsymbol{K}. \quad (98)$$

This data is analyzed.

Step 4. *Choice of size of the priority criterion* of $q \in \boldsymbol{K}$. (Decision - making). On the message: "Enter the size of priority criterion *fq*=" - we enter, the size of the characteristic

$q = 3 \in \boldsymbol{K}$: $f_q = 0.3$.

Step 5. *The relative assessment is calculated.*

For the chosen size of priority criterion f_q=0.3 the relative assessment is calculated:

$$\lambda_q = \frac{f_q - f_q^0}{f_q^* - f_q^0} = \frac{0.30-(-0.16670)}{0.5435-(-0.16670)} = 0.65821. \quad (99)$$

which upon transition from X^o point to X^*_3 lies in the limits:

$$0.35407 = \lambda_3(X^o) \le \lambda_3 = 0.6582 \le \lambda_3(X_3^*) = 1, q \in \boldsymbol{K}$$

Step 6. *Let's calculate coefficient of the linear approximation*

Assuming the linear nature of the change of the criterion of $f_q(X)$ in (11.33) and according to the relative assessment of λ_q, using standard linear approximation techniques, we will calculate the proportionality coefficient between $\lambda_q(X^o)$, λ_q which we will call ρ:

$$\rho = \frac{\lambda_q - \lambda_q(X^o)}{\lambda_q(X_q^*) - \lambda_q(X^o)} = \frac{0.6582-0.35407}{1-0.35407} = 0.4708, q = 3 \in \boldsymbol{K}.$$

Step 7. *Let's calculate coordinates of a priority of the criteria with dimension of* f_q

Assuming the linear nature of change of a vector of $X^q=\{x_1\ \ x_3\}$, q=3 we will determine point coordinates with dimension of fq =0.3, the relative assessment λ_q:

$$X_q = \{x_1^q = x_1^o + \rho(x_q^*(1) - x_1^o), \dots,$$
$$x_N^q = x_N^o + \rho(x_q^*(N) - x_N^o)\}, \quad (100)$$

where $X^o = \{x_1^o, \dots, x_N^o\}$, $X_q^* = \{x_q^*(1), \dots, x_q^*(N)\}$.

$X^o = \{x_1^o$= 2.8039, x_2^o= 30.01}, $X_q^* = \{x_q^*(1)$=3.5, $x_q^*(2)$=12.01}.

As result of the decision (100) we will receive X_q point with coordinates:

$\boldsymbol{X^q} = \{x_1 = 3.1317, x_2 = 21.525\}$.

Step 8. *Calculation of the main indexes of a point of* X^q.

For the received X^q point, we will calculate:

all criteria in the physical units $f_k(X^q) = \{f_k(X^q), k = \overline{1,K}\}$,

$\boldsymbol{f}(\boldsymbol{X^q}) = \{f_1(X^q) = 2.9775, f_2(X^q) = 41.4096, f_3(X^q) = 0.1744, f_4(X^q) = 0.3489\}$;

all relative estimates with the criterion priority

$\lambda^q = \{\lambda_k^q, k = \overline{1,K}\}$, $\lambda_k(X^q) = \frac{f_k(X^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1,K}$,

$\boldsymbol{\lambda_k}(\boldsymbol{X^q}) = \{\lambda_1(X^q) = 0.0369, \lambda_2(X^q) = 0.4258, \lambda_3(X^q) = 0.4811, \lambda_4(X^q) = 0.5189\}$;

minimum relative assessment: $\min(\lambda_k(X^q)) = 0.0369$;

vector of priorities of the third criterion over other criteria:

$P^q(X) = \{p_k^q = \frac{\lambda_q(X^q)}{\lambda_k(X^q)}, k = \overline{1,K}\}$,

$P^q = [p_1^3 = 6.2789, p_2^3 = 17.94, p_3^3 = 1.0, p_4^3 = 0.911]$;

Any point from Pareto's set $\boldsymbol{X_t^o} = \{\lambda_t^o, X_t^o\} \in \boldsymbol{S}^o$ can be similarly calculated.

Analysis of the results of the solution *of the process model represented by the vector problem from the current of symmetry*

In the considered vector problem of nonlinear programming with four heterogeneous criteria – the model of the technological system and the corresponding numerical variant (78)-(82) and built on its basis λ-problems (87)-(92) we obtained the optimum point X^o and the maximum relative estimate λ^o:

$\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 2.8039, x_2 = 30.0, \lambda^o = 0.3541\}\}$;

At the optimum point X^o the criteria in natural units are:

$F(X^o) = \{f_1(X^o) = 3.9345, f_2(X^o) = 77.9319, f_3(X^o) = 0.0844, f_4(X^o) = 0.1687\}$,

and relative units:

$\lambda(X^o) = \{\lambda_1(X^o) = 0.3541, \lambda_2(X^o) = 0.9842, \lambda_3(X^o) = 0.3541, \lambda_4(X^o) = 0.6459\}$;.

This result confirms the proofs of theorem 1 about the most contradictory criteria in the vector optimization problem, i.e.

$\lambda^o = \lambda_1(X^o) = \lambda_3(X^o) = 0.3541, \{1,3\} \in \boldsymbol{K}, X^o \in \boldsymbol{S}.$

The rest of the criteria in relative units lie within:

$\{\lambda_2(X^o) = 0.9842, \lambda_4(X^o) = 0.6459\} > \lambda^o.$

Theoretically, the point $X^o$is the center of symmetry. Indeed, the X^o point lies between the points of the optimum X_1^* and X_3^* obtained for each criterion (step 1), in which $\lambda_1(X_1^*)$=1, $\lambda_3(X_3^*) = 1$.

Let's show geometrically symmetry in Figure 6.

X_1^* $\lambda_1(X_1^*)$=1 X_2^* $\lambda_2(X_2^*)$=1 X_3^* $\lambda_3(X_3^*) = 1$ X_4^* $\lambda_4(X_4^*) = 1$

X^o $\lambda_2(X^o) = 0.9842$

X^o $\lambda_4(X^o) = 0.6459$

X^o $\lambda_1(X^o) = \lambda_3(X^o) = \lambda^o = 0.3541$

Figure 6. Geometric interpretation of symmetry in modeling of technical system with normalized criteria: $\lambda_1(X), \ldots, \lambda_4(X)$.

For the two criteria of the first $\lambda_1(X)$ and the third $\lambda_3(X)$, this example considers an even numerical symmetry.

Analysis of results. The calculated size of criterion $f_q(X_t^o), q{\in}\boldsymbol{K}$ is usually not equal to the set f_q. The error of the choice of $\Delta f_q = |f_q(X_t^o) - f_q| = |0.1744 - 0.3| = 0.125$ is defined by an error of linear approximation, $\Delta f_{q\%} = \frac{\Delta f_q}{f_q} * 100 = 40.2\%$.

In the course of the modeling parametrical restrictions can be changed, i.e. some set of optimum decisions is received. Choose a final version which in our example included from this set of optimum decisions: parameters of technological process $X^o = \{x_1 = 2.8039, x_2 = 30.0\}$;

the parameters of the technological system at a given priority criterion q=3:

$X^q = \{x_1 = 3.1317, x_2 = 21.5248\}$.

3.4.4. Geometrical interpretation of results of the decision in the technological process in three to a measured frame in physical units

We represent these parameters in a two-dimensional x_1, x_2 and three dimensional coordinate system x_1, x_2 and λ in Figures 3, 4, 5, 6 and also in physical units for each function $f_1(X)$, $f_2(X)$, $f_3(X)$, $f_4(X)$ on Figures 7, 8, 9 and 10 respectively.

The first characteristic $f_1(X)$ in physical units show in Figure 7.

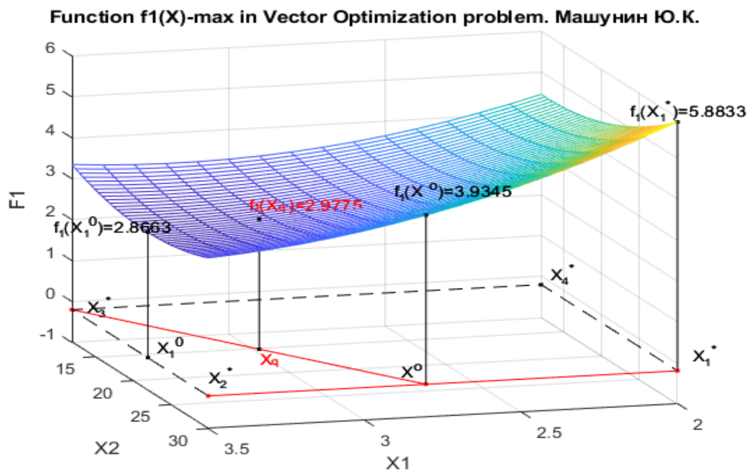

Figure 7. The first characteristics of the $f_1(X)$ technological process in physical unitsin the three-dimensional coordinate system $\{x_1, x_2$ and $f_1(X)\}$.

Optimum points: X_1^* - maximum point;
X^o – Equivalent criteria;
$X_{q=3}$ – priority 3 Criteria;
X_1^0 – Minimum point
are the parameters of the physical characteristics of the technological process.
Let us present the characteristics of the technological process in physical units:
$f_1^* = f_1(X_1^*) \ = \ -5.8834 \leq$

$$f_1(X^o) = 3.9345 \leq$$
$$f_1(X_{q=3}) = 2.9777 \leq$$
$$f_1^0 = f_1(X_1^0) \ = \ 2.8664$$

The second characteristic $\boldsymbol{f_2(X)}$ in physical units show in Figure 8.

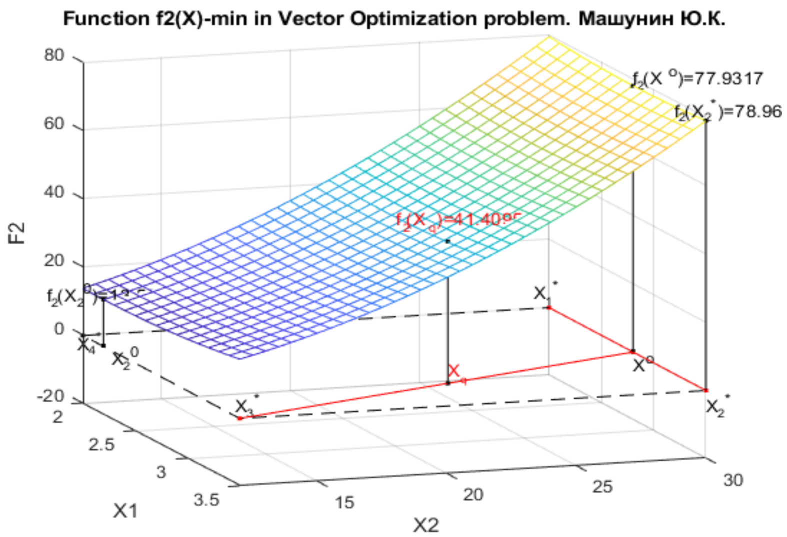

Figure 8. The second characteristics of the $f_2(X)$ technological process in physical unitsin the three-dimensional coordinate system $\{x_1, x_2$ and $f_2(X)\}$

Optimum points: X_2^* - maximum point;
X^o – Equivalent criteria;
$X_{q=3}$ – priority 3 Criteria;
X_2^0 – Minimum point
are the parameters of the physical characteristics of the technological process.
Let us present the characteristics of the technological process in physical units:

$$f_2^* = f_2(X_2^*) = \ 78.931 \leq$$
$$f_2(X^o) = 77.409 \leq$$
$$f_2(X_{q=3}) = 41.409 \leq$$
$$f_2^0 = f_2(X_2^0) \ = 13.561$$

The third characteristic $\boldsymbol{f_3(X)}$ in physical units show in Figure 9.

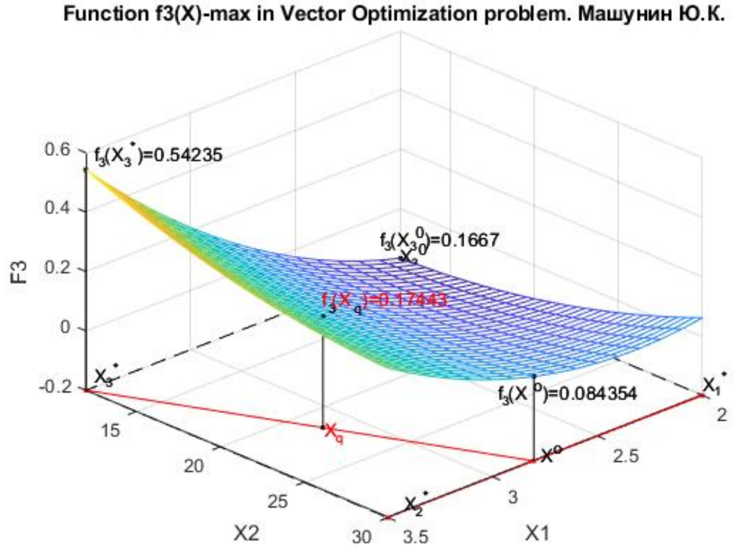

Figure 9. The third characteristics of the $f_3(X)$ technological process in physical unitsin the three-dimensional coordinate system $\{x_1, x_2$ and $f_3(X)\}$

Optimum points: X_3^* - maximum point;
X^o – Equivalent criteria;
$X_{q=3}$ – priority 3 Criteria;
X_3^0 – Minimum point
are the parameters of the physical characteristics of the technological process.
Let us present the characteristics of the technological process in physical units:

$$f_3^* = f_3(X_3^*) = 0.5424 \leq$$
$$f_3(X^o) = 0.0841 \leq$$
$$f_3(X_{q=3}) = 0.3174 \leq$$
$$f_3^0 = f_3(X_3^0) = 0.1667$$

***The fourth characteristic $f_4(X)$** in physical units show in Figure 10.*

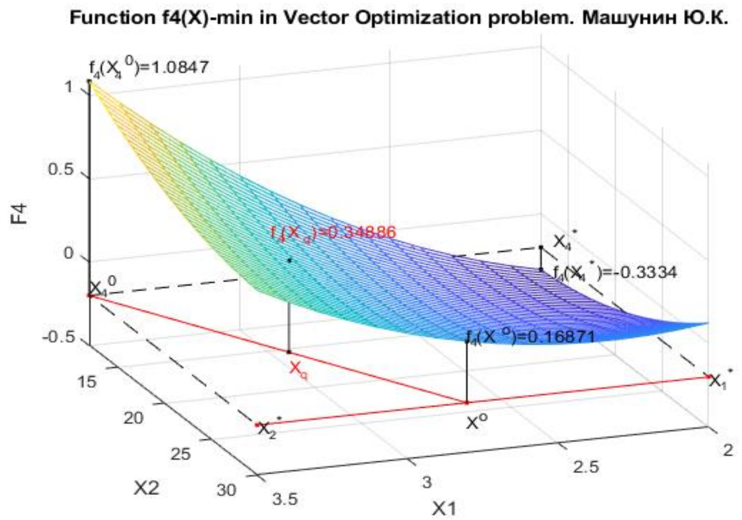

Figure 10. The fourth characteristics of the $f_4(X)$ technological process in physical unitsin the three-dimensional coordinate system $\{x_1, x_2$ and $f_4(X)\}$

Optimum points: X_4^* - maximum point;

X^o – Equivalent criteria;

$X_{q=3}$ – priority 3 Criteria;

X_4^0 – Minimum point

are the parameters of the physical characteristics of the technological process.

Let us present the characteristics of the technological process in physical units:

$$f_4^* = f_4(X_4^*) = 0.3335 \leq$$
$$f_4(X^o) = 0.1687 \leq$$
$$f_4(X_{q=3}) = 0.3468 \leq$$
$$f_4^0 = f_4(X_4^0) = 1.0848$$

$F(X^o) = \{f_1(X^o), f_2(X^o), f_3(X^o), f_4(X^o)\}$;

• relative estimates of $\lambda(X^o) = \{\lambda_1(X^o), \lambda_2(X^o), \lambda_3(X^o), \lambda_4(X^o)\}$;

• maximin λ^o - relative level such that $\lambda^o \leq \lambda_k(X^o)\ \ \forall k \in \boldsymbol{K}$

there is an optimal solution with equivalent criteria (characteristics), and the procedure for obtaining is the adoption of the optimal solution in the process with equivalent criteria (characteristics).

The second option:

• point - $X^q = \{X^{1q}, X^{2q}\}$; characteristics $f(X^q) = \{f_1(X^q), f_2(X^q), f_3(X^q), f_4(X^q)\}$;

• relative estimates $\lambda_k(X^q) = \{\lambda_1(X^q), \lambda_2(X^q), \lambda_3(X^q), \lambda_4(X^q)$;

• maximin λ^{oq} is a relative level such that $\lambda^{oq} \leq \lambda_1(X^q), \lambda_2(X^q), \lambda_3(X^q), \lambda_4(X^q)$;

4. Mathematical Model and Methodology for Modeling and Making an Optimal Decision on the Choice of Parameters of Complex Engineering Systems in Conditions of Certainty, Uncertainty

As an object of research, we consider "Engineering systems", which include "technical systems", "technological processes", "materials", [17, 18 and 19]. The engineering system study is carried out, first, under conditions of certainty, when data on functional characteristics of the engineering system are known; second, in uncertainty conditions where discrete values of individual characteristics are known; there are also known data on limitations imposed on the operation of the system. Mathematical apparatus of engineering system modeling is based on theory and methods of vector optimization, which are presented in the second section. In organizational terms, the process of modeling and simulation of the technical system is presented in the form of a methodology:

"Methodology of selection of optimal parameters of engineering systems in conditions of certainty and uncertainty."

4.1. Types of Problems Arising in the Process of Modeling and Making an Optimal Decision on the Selection of Parameters of Complex Engineering Systems

The problems that arise in the process of making an optimal decision on the selection of optimal parameters of complex Engineering systems on the basis of vector optimization include three types sequentially.

1 type. *Solution of a vector problem of mathematical programming with equivalent criteria*. The result obtained is the basis for further research of the system. In this case, the method of solving a vector problem with equivalent criteria is used. If the result obtained satisfies the decision-maker (decision-maker - designer), then it is taken as a basis. If it does not satisfy, then move on to the second type (direct problem) or the third type of solving vector problems (Inverse problem).

2 type. *The solution of the direct problem of vector optimization*, which consists in the following: "What will be the indicators (characteristics) if the parameters of complex engineering systems are changed." - The method of solving a vector problem with equivalent criteria is used.

3 type. The solution of the inverse problem of vector optimization, which consists in the following: "What will be the parameters of complex engineering systems with given characteristics". - A method for solving a vector problem at a given criterion priority is used.

4.2. Methodology for Modeling and Making an Optimal Decision on the Choice of Parameters of Complex Engineering Systems in Conditions of Certainty, Uncertainty

The methodology is divided into two parts. The first part – "Selection of optimal parameters of engineering systems under conditions of certainty – equivalent criteria": includes two blocks, the second part – "Selection of optimal parameters of engineering systems under conditions of uncertainty – with the priority of one of the criteria": includes two blocks. Each block is divided into a number of stages.

Part 1. The choice of optimal parameters of engineering systems in conditions of certainty are equivalent criteria.

Block 1*. The formation of technical specifications*, the transformation of uncertainty conditions (related to experimental data) into conditions of certainty, the construction of a mathematical and numerical model of an engineering system (the process of modeling of an engineering system) includes 4 stages.

Stage 1. Formation of technical specifications (initial data) for numerical modeling and selection of optimal system parameters. The initial data is formed by the designer who designs the engineering system.

Stage 2. Construction of mathematical and numerical model of the engineering system in conditions of certainty and uncertainty.

Stage 3. Transformation of uncertainty conditions into certainty conditions and construction of a mathematical and numerical model of an engineering system under conditions of certainty.

Stage 4. Construction of an aggregated mathematical and numerical model of an engineering system under conditions of certainty

Block 2. *Making an optimal decision (selection of optimal parameters) in an engineering system with equivalent criteria based on multidimensional mathematics*.

Stage 5. Solution of a vector problem of mathematical programming (VPMP) - a model of an engineering system with equivalent criteria (solution of a direct problem).

Stage 6. Geometric interpretation of the results of the vector problem of mathematical programming solution with *N* parameters and *K* criteria into a two-dimensional coordinate system in relative units.

Stage 7. Analysis of the results of solving the Vector Problem of Mathematical Programming with equivalent criteria is the preparation of information for decision-making with the priority of a separate criterion from the entire set of criteria.

Part 2. Selection of optimal parameters of engineering systems under conditions of uncertainty - with the priority of one of the criteria.

Block 3. Analysis, comparison of the results of the VPMP solution with equivalent criteria of a complex engineering system and preparation for the selection of optimal parameters according to the priority criterion.

Stage 8. Analysis and comparison of the results of solving a vector problem with equivalent criteria with four and two variables. Conclusion from the analysis: ***Information about the Engineering System in physical units*** for decision-making with the priority of the criterion. Preparation for geometric interpretation of the solution of the vector problem of choosing optimal parameters according to the priority criterion.

Stage 9. Geometric interpretation of the solution results in the design of an engineering system for the transition from two-dimensional to *N*-dimensional space in relative units.

Block 4. Selection of optimal parameters of a complex engineering system according to a priority criterion, geometric interpretation of the solution results based on multidimensional optimization.

The selection of the optimal parameters of a complex engineering (technical) system and the geometric interpretation of the solution results according to the priority criterion corresponds to the number of criteria *K*=4: Block *k*1, Block *k*2, Block *k*3, Block *k*4.

Name of the block.

Block ***k*1, ..., Block *k*4**. Selection of optimal parameters of the engineering system (technical system) according to the (first, ..., fourth) priority criterion, geometric interpretation of the results of the VPMP solution.

Each parameter selection block for the corresponding criterion includes the following sections.

1.k1, …, 1.k4. Solution of a vector problem - a model of a complex engineering system (technical system) with a given priority of the first criterion, including:

Solving a vector problem of mathematical programming - a model of an engineering system with equivalent criteria (Step 1).

Select the priority criterion $q \in K$ (Step 2).

Select the numerical value of the selected priority criterion $q \in K$ (Steps 3, 4).

Solutions to a vector problem of mathematical programming - a model of an engineering system with a given priority of the q∈K criterion (Steps 5−8).

2.k1, …, 2.k4.. Analysis of the results of solving a vector problem of mathematical programming - a model of an engineering system at a given priority of the first criterion.

3.k1, …, 3.k4. . Geometric interpretation of the results of solving the vector problem of choosing optimal parameters according to the first priority criterion in relative units.

4.k1, …, 4.k4. Geometric interpretation of the results of the VPMP solution with the priority of the first criterion – the model of the engineering system when designing in a three-dimensional coordinate system *in physical units*.

The other three blocks are formed by analogy.

1. Geometric interpretation of the results of the solution of the first criterion with the priority of the first criterion – the model of the engineering system when designing in a three-dimensional coordinate system *in physical units*. **…**

4. Geometric interpretation of the results of the solution of the VZMP of the fourth criterion with the priority of the first criterion - the model of the engineering system when designing in a three-dimensional coordinate system *in physical units*.

5. Research, modeling and selection of optimal parameters of a technical system under conditions of certainty and uncertainty on the basis of multidimensional mathematics.

5.0 Modeling and Making an Optimal Decision on a Numerical Set of Criteria of a Technical System with Four Parameters

As an object of research, we consider "Engineering systems", which include "Technical System" [18, 20-26]. The structure of the material is considered with four parameters and four characteristics. Experimental data of the material structure are presented in the form of a decision-making problem of the second type (70):

As an object of research, we consider "Engineering systems", which include "technical system", [18, 20-26]. The technical system is considered with four parameters and four characteristics. The experimental data of the technical system are presented in the form of a decision-making problem:

$$FF = \left| \begin{matrix} x_{1,1}\ x_{2,1}\ x_{3,1}\ x_{4,1}\ \ f_1^1, \dots, f_1^K \\ \dots \\ x_{1,M}\ \ x_{2,M}\ \ x_{3,M}\ x_{4,M}\ f_M^1, \dots, f_M^K \end{matrix} \right| \text{ - problem of 2-type, } K = 4.$$

The study of the engineering system was carried out:

firstly, under conditions of certainty, when the data on the functional characteristics of the engineering system are known; secondly, under conditions of uncertainty, when the discrete values of individual characteristics are known; There is also evidence of the constraints that are imposed on the functioning of the system.

Using regression analysis, the experimental data F is converted into a vector problem of mathematical programming:

$$max\ f_k(X,A) = a_{0k} + a_{1k}x_1 + a_{2k}x_2 + a_{3k}x_3 + a_{4k}x_4 + a_{5k}x_1x_2 + a_{6k}x_1x_3 + a_{7k}x_1x_4 + a_{8k}x_2x_3 + a_{9k}x_2x_4 + a_{10k}x_3x_4 + a_{11k}x_1^2 + a_{12k}x_2^2 + a_{13k}x_2^2 + a_{14k}x_2^2,\ k = \overline{1,\ K_1}.$$

$$min\ f_k(X,A) = a_{0k} + a_{1k}x_1 + a_{2k}x_2 + a_{3k}x_3 + a_{4k}x_4 + a_{5k}x_1x_2 + a_{6k}x_1x_3 + a_{7k}x_1x_4 + a_{8k}x_2x_3 + a_{9k}x_2x_4 + a_{10k}x_3x_4 + a_{11k}x_1^2 + a_{12k}x_2^2 + a_{13k}x_2^2 + a_{14k}x_2^2,\ k = \overline{1,\ K_2}.$$

Under the constraints of $x_1^0 \le x_1 \le x_1^*, x_2^0 \le x_2 \le x_2^*, x_3^0 \le x_3 \le x_3^*, x_4^0 \le x_4 \le x_4^*$.

$x_1 + x_2 + x_3 + x_4 = a.$

The mathematical apparatus of modeling an engineering system is based on the theory and methods of vector optimization, which are presented in the previous chapters.

In organizational terms, the process of modeling and simulation of an engineering system is presented in the form of a methodology: "Methodology for choosing the optimal parameters of engineering systems under conditions of certainty and uncertainty", presented in section 4.

Part 1. The choice of optimal parameters of engineering systems in conditions of certainty are equivalent criteria.

5.1. Block 1. The technical assignment: Construction of a mathematical and numerical model of the technical system of a complex structure.

The first stage, as well as the stage of analyzing the results of the solution, choosing the priority criterion and its value, is performed by ***the constructor of the technical system*** of a complex structure. The remaining stages are performed by ***a mathematician-programmer***.

5.1.1. ***Stage 1****.* ***Terms of reference****: "Selection of optimal parameters of the technical system of complex structure".*

It is given. The functioning of a technical system (TS) is determined by four parameters $X = \{x_1, x_2, x_3, x_4\}$, which represent a vector of controlled variables. The parameters of the vehicle are set within the following limits:

$$21 \le x_1 \le 79,\ 5 \le x_2 \le 59,\ 2.1 \le x_3 \le 9,\ 2.2 \le x_4 \le 7. \qquad (101)$$

The functioning of the vehicle is determined by four characteristics (criteria) $F(X) = \{f_1(X), f_2(X), f_3(X), f_4(X)\}$, the value of which depends on the vector of the parameters

$X = \{x_j, j = \overline{1,N}, N = 4\}$.

Conditions of certainty. They are characterized by the fact that for the first characteristic $h_1(Y)$ the functional dependence on the parameters $X = \{x_j, j = \overline{1,N}, N = 4\}$. is known.

$$f_1(X) = 323.84 - 2.249x_1 - 3.49x_2 + 10.7267x_3 + 13.124x_4 + 0.0968x_1x_2 - 0.062x_1x_3 - 0.169x_1x_4 + 0.0743x_2x_3 - 0.1042x_2x_4 - 0.0036x_3x_4 + 0.0143x_1^2 + 0.0118x_2^2 - 0.2434x_3^2 - 0.5026x_4^2, \qquad (102)$$

Uncertainty conditions. They are characterized by the fact that for the second, third, and fourth characteristics: $f_2(X), f_3(X), f_4(X)$ the results of experimental data are known: the values of the parameters and the corresponding characteristics.

Numerical values of parameters X and characteristics of $f_2(X), f_3(X), f_4(X)$ are presented in table 1.

Table 1.

Numerical values of components and characteristics of the technical system.

x_1	x_2	x_3	x_4	$f_2(X)$	$f_3(X)$	$f_4(X)$
20	0	2	2	1149.6	115.1	24.24 27.60
20	0	2	5	1164.0	114.5	28.80 30.00
20	0	2	8	1176.0	114.4	31.20 32.40
20	0	5	2	1212.0	118.8	33.60 34.80

20	0	5	5	1260.0	113.8	34.80 19.92
20	0	5	8	1257.6	113.3	21.60 25.20
20	0	8	2	1256.4	110.7	29.76 33.48
20	0	8	5	1252.8	109.2	37.20 39.48
20	0	8	8	1251.6	108.5	42.00 49.20
20	30	2	2	2143.2	128.3	15.60 18.00
20	30	2	5	2154.0	127.4	21.60 24.24
20	30	2	8	2163.6	126.8	28.80 32.40
20	30	5	2	2176.8	126.1	35.16 39.60
20	30	5	5	2185.2	124.3	44.88 11.28
20	30	5	8	2198.4	124.1	14.40 16.80
20	30	8	2	2211.6	123.9	21.12 22.80
20	30	8	5	2232.0	121.4	27.60 30.84
20	30	8	8	2245.2	121.7	36.00 40.56
20	60	2	2	2954.4	150.4	52.80 60.00
20	60	2	5	2820.0	144.9	64.80 68.64
20	60	2	8	2772.0	140.8	75.60 82.80
20	60	5	2	2748.0	138.6	88.08 97.20
20	60	5	5	2832.0	140.8	107.64
20	60	5	8	2904.0	143.5	40.56 45.60
20	60	8	2	3022.8	146.0	52.80 60.00
20	60	8	5	3036.0	144.9	67.20 73.20
20	60	8	8	3056.4	143.8	79.44 85.20
50	0	2	2	3583.2	181.3	99.00 31.92
50	0	2	5	3601.2	180.8	36.00 43.20
50	0	2	8	3608.4	179.4	51.36 61.20
50	0	5	2	3616.8	179.1	72.00 82.80
50	0	5	5	3622.8	178.0	86.40 90.36
50	0	5	8	3637.2	177.6	23.28 30.00
50	0	8	2	3651.6	176.9	36.00 42.72
50	0	8	5	3672.0	175.3	48.00 54.00
50	0	8	8	36852	174.7	62.16 73.20
50	30	2	2	1195.2	123.6	81.72 87.00
50	30	2	5	1212.0	118.7	94.80
50	30	2	8	1236.0	115.9	103.20
50	30	5	2	1251.6	115.1	116.16
50	30	5	5	1272.0	113.2	126.00
50	30	5	8	1296.0	111.8	136.80
50	30	8	2	1318.8	110.7	145.44
50	30	8	5	1344.0	108.2	156.00
50	30	8	8	1388.4	106.3	174.72
50	60	2	2	2176.8	132.8	
50	60	2	5	2196.0	131.1	
50	60	2	8	2220.0	129.7	
50	60	5	2	2245.2	128.3	
50	60	5	5	2286.0	127.0	
50	60	5	8	2294.4	125.6	
50	60	8	2	2313.6	123.9	

50	60	8	5	2340.0	114.5	
50	60	8	8	2382.0	119.5	
80	0	2	2	2988.0	154.8	
80	0	2	5	3012.0	153.2	
80	0	2	8	3036.0	151.8	
80	0	5	2	3056.4	150.4	
80	0	5	5	3108.0	150.7	
80	0	5	8	3156.0	151.2	
80	0	8	2	3244.8	151.5	
80	0	8	5	3228.0	144.9	
80	0	8	8	3193.2	140.8	
80	30	2	2	3616.8	185.7	
80	30	2	5	3639.6	183.5	
80	30	2	8	3660.0	182.2	
80	30	5	2	3685.2	181.3	
80	30	5	5	3708.0	179.4	
80	30	5	8	3732.0	178.0	
80	30	8	2	3753.6	176.9	
80	30	8	5	3672.0	175.3	
80	30	8	8	3822.0	172.5	
80	60	2	2	1218.0	128.3	
80	60	2	5	1248.0	125.6	
80	60	2	8	1272.0	124.2	
80	60	5	2	1318.8	121.7	
80	60	5	5	1344.0	118.7	
80	60	5	8	1392.0	115.9	
80	60	8	2	1422.0	115.1	
80	60	8	5	1464.0	110.4	
80	60	8	8	1524.0	108.5	
min $f_i(X)$, i=1,…, 81				1149.6	92.4	11.3
max $f_i(X)$, i=1,…, 81				3822.0	161.5	174.7

On the parameters $X = \{x_1, x_2, x_3, x_4\}$ are in total imposed restriction:
$x_1 + x_2 + x_3 + x_4 = 102.$

In the made decision, assessment size on the first and third characteristic $f_1(X), f_3(X)$ it is desirable (criterion), to receive as above $f_1(X) \rightarrow max\ f_3(X) \rightarrow max$, on second and fourth characteristic $f_2(X), f_4(X)$ as low as possible is possible $f_2(X) \rightarrow min\ f_4(X) \rightarrow min$.

In the pilot studies the rate of an increase of parameters:
$X = \{x_1, x_2, x_3, x_4\}$ will be defined in the following limits:

$$x_1 \in [20.\ 50.\ 80.];\ x_2 \in [0.\ 30.\ 60.];\ x_3 \in [2.\ 5.\ 8.],\ x_4 \in [2.\ 5.\ 8.]. \quad (103)$$

It is required. 1) To construct mathematical model of the technical system in the form of a VPMP. 2) To carry out model operation: first, on the basis of the constructed mathematical model of a vector problem, secondly, on the basis of methods of solution of a VPMP of non-linear programming at equivalent criteria, and, thirdly, the software developed for these purposes in the MATLAB system. To make an optimal solution: The choice of optimum composition of the technical system according to its functional characteristics taking into account their equivalence. 4) Choose a priority criterion. Solve the vector optimization problem and make the best (optimal) solution with the given priority of the criterion. 5) To present a geometric interpretation of the results of the solution in the design of an engineering system for the transition from two-dimensional to N-dimensional space in relative units. 6) To show the geometric interpretation of the solution results in the design of an engineering system for the transition from two-dimensional to *N*-dimensional space in physical units.

Note: The author has developed software in the MATLAB system for solving vector problems of mathematical programming. The vector problem includes four variables (parameters of the technical system): $X = \{x_1, x_2, x_3, x_4\}$ and four criteria (characteristics)

$F(X) = \{f_1(X), f_2(X), f_3(X), f_4(X)\}$. But for each new data (new system), the program is configured individually. In software, the criteria are $F(X) = \{f_1(X), \dots, f_4(X)\}$ uncertainty conditions (Table 1).

Stage 1a. Mathematical model of the technical system under conditions of certainty

The construction of a mathematical model for making an optimal managerial decision on the structure of the technical system is shown in Section 1. In accordance with (5a)-(10a), let us present a mathematical model of the technical system l under conditions of certainty and uncertainty in the aggregate in the form of a vector problem:

$$Opt\ F(X) = \{\max F_1(X) = \{\max f_k(X), k = \overline{1, K_1^{def}}\}, \quad (104)$$

$$\max I_1(X) \equiv \{\max f_k(X_i,\ i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}, \quad (105)$$

$$\min F_2(X) = \{\min f_k(X), k = \overline{1, K_2^{def}}, \quad (106)$$

$$\min I_2(X) \equiv \{\min f_k(X_i,\ i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}\}, \quad (107)$$

at restrictions: $f_k^{min} \le f_k(X) \le f_k^{max}, k = \overline{1, K}$,

$$x_j^{min} \le x_j \le x_j^{max}, j = \overline{1, N}, x_1 + x_2 + x_3, +x_4 = a, \quad (108)$$

where $X = \{x_j,\ j = \overline{1, N}\}$- vector of controlled variables (structural parameters of the technical system);

$F(X) = \{F_1(X)\ F_2(X)\ I_1(X)\ I_2(X)\}$ represents a vector criterion, each component of which is a vector of criteria (characteristics) of the technical system, which are functionally dependent on the discrete values of the vector of variables;

$F_1(X) = \{f_k(X), k = \overline{1, K_1^{def}}\}$, $F_2(X) = \{f_k(X), k = \overline{1, K_2^{def}}\}\}$ is a set of *max* and *min* functions, respectively;

$I_1(Y) = \{\{f_k(X_i, i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}$,

$I_2(Y) = \{\{f_k(X_i, i = \overline{1, M})\}^T, k = \overline{1, K_2^{unc}}\}$ is the set of matrices *max* and *min*, respectively; (d*efiniteness*) K_1^{unc}, K_2^{unc} (*uncertainty*) a set of criteria max and min formed under conditions of certainty and uncertainty;

in (108) $f_k^{min} \le f_k(X) \le f_k^{max}, k = \overline{1, K}$ is represented by the vector function of the constraints imposed on the functioning of the technical system;

in (108) $x_j^{min} \le x_j \le x_j^{max}, j = \overline{1, N}$ – parametric constraints.

Let's assume that the functions $f_k(X), k = \overline{1, K}$ are differentiable and convex, $g_i(X),\ i = \overline{1, M}$ are continuous, and the set of valid points ***S*** given by constraints (108) is not empty:

$\boldsymbol{S} = \{X \in R^n | G(Y) \le 0, X^{min} \le X \le X^{max}\} \ne \emptyset$**,** and is a compact.

5.1.2. ***Stage 2.*** *Numerical Model of the technical system under Conditions of Certainty*

Certainty conditions are characterized by the functional dependence of each characteristic and constraint of the technical system. In the example, the characteristic (104) and constraints (108) are known. Using the information data (101), (102) we construct a single-criteria problem of nonlinear programming under conditions of certainty:

$$max\ f_1(X) = 323.84 - 2.249x_1 - 3.49x_2 + 10.7267x_3 + 13.124x_4 + 0.0968x_1x_2 - 0.062x_1x_3 - 0.169x_1x_4 + 0.0743x_2x_3 - 0.1042x_2x_4 - 0.0036x_3x_4 + 0.0143x_1^2 + 0.0118x_2^2 - 0.2434x_3^2 - 0.5026x_4^2, \quad (109)$$

at restrictions: $x_1 + x_2 + x_3 + x_4 = 102$,

$21 \le x_1 \le 79, 5 \le x_2 \le 59, 2.1 \le x_3 \le 9.0, 2.2 \le x_4 \le 7.0.$ (110)

This data is further used to build a mathematical model of the technical system.

*5.1.3. **Stage 3**. Transformation of experimental data (uncertainty conditions) into data with functional dependence (certainty conditions)*

Conditions of uncertainty are characterized by the fact that the initial data characterizing the object under study are represented by: a) random, b) fuzzy, or c) incomplete data, i.e., under conditions of uncertainty, only a finite set of measured parameters $X = \{x_j, j = \overline{1,N}\}$ are known:

$X_v = \{x_{iv}, v = \overline{1,V}\}, i = \overline{1,M}$, where $v = \overline{1,V}$ - the number of parameters that are studied in the model of the technical system, $i = \overline{1,M}$- number and set of data; A set of K characteristics of a technical system: $f_k(X_v = \{x_{iv},\ v = \overline{1,V}\}, i = \overline{1,M}), k = \overline{1,K}$.

Therefore, under conditions of uncertainty, there is not enough information about the functional dependence of each characteristic and constraints on the parameters. The information data of options a) and b) shall be converted into numerical data of option c) and shall be presented in tabular form. The paper considers option c) information with incomplete data, which, as a rule, is obtained as ***a result of an experiment***.

Taking into account the measured parameters, X_v and the corresponding set of K characteristics: $f_k(X_v = \{x_{iv},\ v = \overline{1,V}\}, i = \overline{1,M}), k = \overline{1,K}$. Let us present a matrix of the results of experimental data on the technical system under study:

$$I = \begin{vmatrix} a_1 \\ \dots \\ a_M \end{vmatrix} = \begin{vmatrix} X_1 = x_{11}, x_{12}, x_{13}, x_{14}\ f_2(X_1), f_3(X_1), f_4(X_1) \\ \dots \\ X_M = x_{M1}, x_{M2}, x_{M3}, x_{M4} f_2(X_M), f_3(X_M), f_4(X_M) \end{vmatrix}, \quad (111)$$

Let us present a mathematical model of the technical system under uncertainty in the form of a vector problem of mathematical programming:

$$Opt\ F(X) = \{\max I_1(Y) \equiv \{\max f_k\ (X_i,\ i = \overline{1,M})\}^T, k = \overline{1, K_1^{unc}}\}, \quad (112)$$

$$\min I_2(Y) \equiv \{\min f_k\ (X_i,\ i = \overline{1,M})\}^T, k = \overline{1, K_2^{unc}}\}\}, \quad (113)$$

$$\text{at restriction} \quad f_k^{min} \le f_k(X) \le f_k^{max}, k = \overline{1,K}, \quad (114)$$

$$\textstyle\sum_{v=1}^{V} x_v(t) = a,\ x_v^{min} \le x_v \le x_v^{max}, v = \overline{1,V}, \quad (115)$$

where $X = \{x_v,\ v = \overline{1,V}\}$ is the vector of controlled variables (parameters);

$F(X) = \{I_1(X)\ I_2(X)\}$ is a vector criterion, each component of which represents a vector of criteria (output characteristics of the object under study). The value of the characteristic (function) depends on the discrete values of the vector of the variables X. $I_1(X) = \overline{1, K_1^{unc}}$, $I_2(X) = \overline{1, K_2^{unc}}$ (*uncertainty*) – a set of max and min criteria formed under uncertainty;

in (114) $f_k^{min} \le f_k(X) \le f_k^{max}, k = \overline{1,K}$ – vector-function of the restrictions imposed on the functioning of the object under study, $x_v^{min} \le x_v \le x_v^{max},\ v = \overline{1,V}$ – parametric constraints of the object under study.

*5.1.4. **Stage 4**. Construction of an Aggregated Mathematical and Numerical Model of the Structure of the Material under Uncertainty.*

Formation under uncertainty consists in the use of qualitative, quantitative descriptions of the technical system, obtained according to the principle of "input-output" in Table 2.

Transformation of initial data (information): $f_2(X_i, i = \overline{1,M})$, $f_3(X_i, i = \overline{1,M})$, $f_4(X_i, i = \overline{1,M})$ in functional form: $f_2(X), f_3(X), f_4(X)$ is carried out by using mathematical methods of regression analysis.

The initial data are generated in Table 2 in the MATLAB system in the form of a matrix:

$$I = |X, F| = \{x_{i1}, x_{i2}, x_{i3}, x_{i4}, f_{i2},\ f_{i3},\ f_{i4}, i = \overline{1,M}\}. \quad (116)$$

For each set of experimental data in Table 2 $f_k, k = 2,3,4$ the regression function is constructed by the least squares method $\mathbf{min} \sum_{i=1}^{M}(x_i - \overline{x_i})^2$ in the MATLAB system.

A polynomial A_k is formed, which determines the relationship between the parameters $X_i = \{x_{i1}, x_{i2}, x_{i3}, x_{i4}\}$ and the $\overline{f_{ki}} = f(X_i, A_k), k = 2, 3, 4$.

The result of the formation is a system of coefficients:

$A_k = \{A_{0k}, A_{1k}, \dots, A_{14k}\}$, that define the coefficients of a quadratic polynomial:

$f_k(X, A) = A_{0k} + A_{1k}x_1 + A_{2k}x_2 + A_{3k}x_3 + A_{4k}x_4 + A_{5k}x_1x_2 + A_{6k}x_1x_3 + A_{7k}x_1x_4 + A_{8k}x_2x_3 + A_{9k}x_2x_4 + A_{10k}x_3x_4 + A_{11k}x_1^2 + A_{12k}x_2^2 + A_{13k}x_3^2 + A_{14k}x_4^2, k = 2, 3, 4.$

Polynomial approximation software with four variables and fourteen factors has been developed. As a result, the experimental data of Table 2 are transformed by the system of coefficients of the three functions $f_k(X, A)$ in the form of a table (Program: Z_Material_MMTT32_os13_4k):

Ao=[323.8408 954.8634 110.02 21.0051 (117)
-2.2495 28.6719 0.9106 -0.0101
-3.4938 37.0392 0.6206 -0.8403
10.7267 -31.0303 -0.4287 -0.4314
13.1239 -54.0031 -2.5176 1.1718
0.0969 -0.9219 -0.0151 0.0166
-0.0621 0.5644 -0.0094 0.0850
-0.1696 0.8966 0.0222 -0.0001
0.0743 -0.1540 -0.0198 0.0522
-0.1042 0.3919 0.0184 0.0003
0.0036 -0.0135 -0.0006 0.0006
0.0142 0.0477 -0.0004 -0.0021
0.0117 0.0437 -0.0003 0.0035
-0.2433 3.8489 0.0390 0.0061
-0.5026 3.1748 0.1414 -0.0310];
Rj =[0.6115 0.7149 0.6551 0.9017];
RRj=[0.3740 0.5111 0.4292 0.8130];

On the basis of Ao(2) Ao(3) Ao(4), the functions $f_2(X)$, $f_3(X)$ and $f_4(X)$ are constructed, which, taking into account the obtained coefficients, will take the form:

$$\max f_3(X), \min f_2(X), \min f_4(X), \text{при ограничениях: (110).} \quad (118)$$

Minimum and maximum values of experimental data are $x_1, \dots, x_4$. are presented at the bottom of Table 2. The minimum and maximum values of the functions $f_1(X), f_2(X), f_3(X)$ *and* $f_4(X)$ differ slightly from the experimental data.

The correlation index and coefficients of determination are presented in the lower rows of Table 2. The results of regression analysis (99)-(103) are further used in the construction of a mathematical model of the technical system.

The resulting mathematical model in the form of VPMP (117), (118) is a numerical model of the structure of the technical system under uncertainty (Numerical data of the model (118) are presented below).

5.1.5. Construction of an aggregated numerical model of the structure of the technical system under conditions of certainty

Combining mathematical models of the technical system under conditions of certainty (109)-(110) and uncertainty (118), we will present a mathematical model of the technical system under conditions of certainty and uncertainty in the aggregate in the form of a vector problem:

$$Opt\ F(X) = \{max\ F_1(Y) = \{max\ f_k\ (X), k = \overline{1, K_1^{def}}\}, \quad (119)$$
$$max\ I_1(Y) \equiv \{max\ f_k\ (X_i,\ i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\},$$
$$min\ F_2(Y) = \{min\ f_k\ (X), k = \overline{1, K_2^{def}},$$
$$min\ I_2(X) \equiv \{min\ f_k\ (X_i,\ i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}\},$$

at restriction $f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1,K}, x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1,N}.$

where $X = \{x_j,\ j = \overline{1,N}\}$ - vector of controlled variables (design parameters);

$F_1(X) = \{f_k(X), k = \overline{1, K_1^{def}}\}, F_2(X) = \{f_k(X), k = \overline{1, K_2^{def}}\}$ –

a set of *max* and *min* features respectively;

$I_1(Y) = \{\{f_k(X_i, i = \overline{1,M})\}^T, k = \overline{1, K_1^{unc}}\}, I_2(X) = \{\{f_k(Y_i, i = \overline{1,M})\}^T, k = \overline{1, K_2^{unc}}\}$

a set of *max* and *min* matrices, respectively; (*definiteness*), $K_1^{unc}. K_2^{unc}$ (*uncertainty*) is a set of criteria *max* and *min* formed under conditions of certainty and uncertainty. Combining the numerical models of the technical system under conditions of certainty (116) and uncertainty (118), let us present the numerical model of the technical system under conditions of certainty and uncertainty in the aggregate in the form of a vector problem:

$$opt\ F(X) = \{\max F_1(X) = \{\max f_1(X) \equiv 323.84 - 2.25x_1 - 3.49x_2 + 10.72x_3 + 13.124x_4 + 0.0968x_1x_2 - 0.062x_1x_3 - 0.169x_1x_4 + 0.0743x_2x_3 - 0.1x_2x_4 - 0.0036x_3x_4 + 0.0143x_1^2 + 0.0118x_2^2 - 0.2434x_3^2 - 0.5026x_4^2, \quad (120)$$

$$max\ f_3(X) \equiv 110.22 + 0.7918x_1 + 1.73x_2 - 0.3713x_3 - 2.20x_4 - 0.0132x_1x_2 - 0.008x_1x_3 + 0.0193x_1x_4 - 0.0172x_2x_3 + 0.0161x_2x_4 - 0.0006x_3x_4 - 0.0004x_1^2 - 0.0002x_2^2 + 0.0335x_3^2 + 0.124x_4^2\}, \quad (121)$$

$$\text{Min}\ F_2(X) = \{\min f_2(X) \equiv 954.86 + 28.67x_1 + 37.03x_2 - 31.03x_3 + 54x_4 - 0.922x_1x_2 - 2x_1x_3 + 0.896x_1x_4 - 0.154x_2x_3 + 0.3919x_2x_4 - 0.0134x_3x_4 + 0.0478x_1^2 + 0.0438x_2^2 + 3.8489x_3^2 + 3.1748x_4^2, \quad (122)$$

$$\max f_4(X) \equiv 21.004 - 0.0097x_1 - 0.841x_2 - 0.4326x_3 + 1.1723x_4 + 0166x_1x_2 + 0.085x_1x_3 - 0.0001x_1x_4 + 0.0523x_2x_3 + 0.0002x_2x_4 + 0.0006x_3x_4 - 0.0022x_1^2 + 0.0035x_2^2 + 0.006x_3^2 - 0.0311x_4^2\}\}, \quad (123)$$

$$\text{at restrictions: } x_1 + x_2 + x_3 + x_4 = 102, \quad (124)$$

$$21 \leq x_1 \leq 79,\ 5 \leq x_2 \leq 59, 2.10 \leq x_3 \leq 9.0, 2.2 \leq x_4 \leq 7.0. \quad (125)$$

The VPMP (120)-(125) represents the model of optimal decision-making, i.e. the choice of the optimal of the technical system in conditions of certainty and uncertainty in the total.

5.1.6. Block 2. Making an optimal decision (selection of optimal parameters) of a technical system based on a vector problem of mathematical programming.

Stage 5. *Solution of a vector problem of mathematical programming (VPMP) - a model of an engineering system with equivalent criteria (solution of a direct problem).*

In order to form model of the technical system on the basis of the vector problem of mathematical programming with equivalent criteria (120)-(125), methods based on the axiomatic normalization of criteria and the principle of guaranteed result, resulting from axiom 1 and the principle of optimality 1, are presented. The solution methodology is presented in the form of a series of steps.

Step 1. Decides a vector problem of mathematical programming (120)-(125) by each criterion separately, at the same time the function *fmincon*(…) of the MATLAB system is used, the appeal to the function *fmincon*(…) is considered in [18, 19]. As a result of calculation VPMP for each criterion we receive optimum points: X_k^* and $f_k^* = f_k(X_k^*), k = \overline{1,K}$ $\boldsymbol{K} = 4$ sizes of criteria in this point:

Criterion 1: $X_1^* = \{x_1 = 47.6032, x_2 = 44.1968, x_3 = 8, x_4 = 2.2\}, f_1^* = f_1(X_1^*) = -392.9$;

Criterion 2: $X_2^* = \{x_1 = 56.6036, x_2 = 35.1964, x_3 = 8, x_4 = 2.2\}, f_2^* = f_2(X_2^*) = 1337.4$;

Criterion 3: $X_3^* = \{x_1 = 33.9000, x_2 = 59, x_3 = 2.1, x_4 = 7\}, f_3^* = f_3(X_3^*) = -210.5729$;

Criterion 4: $X_4^* = \{x_1 = 37.7000, x_2 = 59, x_3 = 2.1, x_4 = 2.2\}, f_4^* = f_4(X_4^*) = 32.68$.

The result the solution of a VPMP of non-linear programming (120)-(125) the optimum points $X_1^*, X_2^*, X_3^*, X_4^*$ in three-dimensional frames of x_1, x_3 and f_1, f_2, f_3, f_4 is presented on Fig. 11.

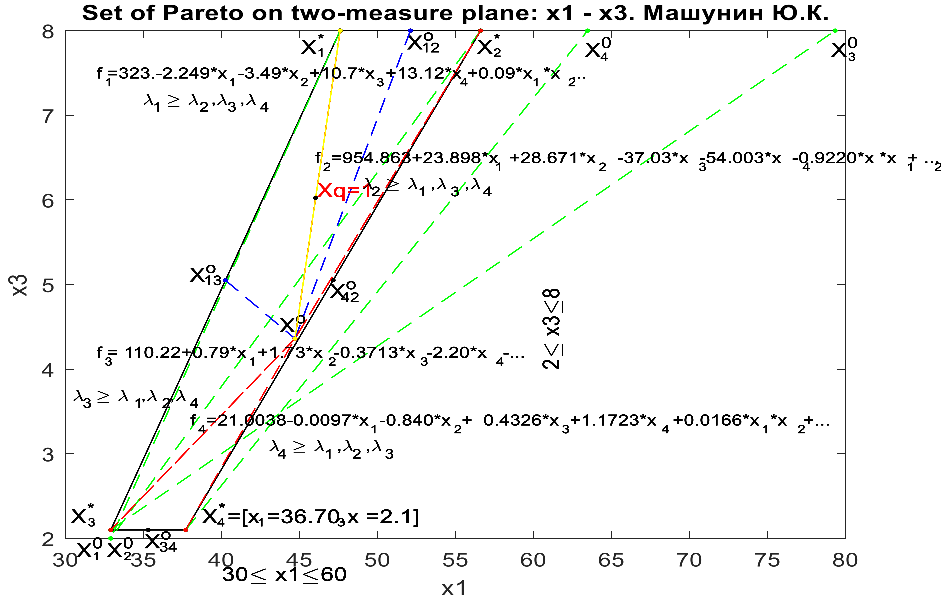

Figure 11. The permissible set of points ***S*** in the coordinates $\{x_1, x_3\}$. A set of Pareto-optimal points ***S***^{o}⊂***S***, bounded by the points X_1^*, X_2^*, X_3^*, X_4^* and the local $\boldsymbol{S}_1^o$ between $X_1^*, X_{13}^{o*}, X^o, X_{12}^o, X_1^*$. in the two-dimensional coordinate system $\{x_1, x_3\}$.

Step 2. The worst unchangeable part of each criterion of the VPMP (120)-(125) is defined (anti-optimum) (A superscript zero):

$X_1^0 = \{x_1 = 32.90, x_2 = 60.0, x_3 = 2.1, x_4 = 7.00\}, f_1^0 = f_1(X_1^0) = 302.76;$

$X_2^0 = \{x_1 = 32.90, x_2 = 60.0, x_3 = 2.1, x_4 = 7.0\}, f_2^0 = f_2(X_2^0) = -2417.52$

$X_3^0 = \{x_1 = 78.999, x_2 = 10.4047, x_3 = 8, x_4 = 4.5953\}, f_3^0 = f_3(X_3^0) = 170.53;$

$X_4^0 = \{x_1 = 63.465, x_2 = 24.5125, x_3 = 8, x_4 = 6.022\}, f_4^0 = f(X_4^0) = -75.205.$

Step 3. Systems analysis of a set of points in the vector problem of mathematical programming that are Pareto-optimal is performed, (i.e., the analysis by each criterion). In points of an optimum of $X^* = \{X_1^*, X_2^*, X_3^*, X_4^*\}$ sizes of target functions of

$F(X^*) = \|f_q(Y_k^*)\|_{q=\overline{1,K}}^{k=\overline{1,K}}$, a vector of $D = (d_1\ d_2\ d_3\ d_4)^T$ of deviations are determined by each criterion on an admissible set of ***S***: $d_k = f_k^* - f_k^0, k = \overline{1,4}$,

and matrix of the relative estimates of $\lambda(Y^*) = \|\lambda_q(Y_k^*)\|_{q=\overline{1,K}}^{k=\overline{1,K}}$, where $\lambda_k(X) = (f_k^* - f_k^0)/d_k$

$d_k = \{92.72, -1114.9, 41.2898, -43.1973\}.$

$$F(X^*) = \left\|\begin{matrix} 392.9 & 1419.5 & 185.0 & 70.4 \\ 387.2 & 1337.4 & 178.5 & 73.8 \\ 302.8 & 2417.5 & 210.6 & 32.2 \\ 338.2 & 2162.2 & 208.0 & 32.7 \end{matrix}\right\|, \lambda(X^*) = \left\|\begin{matrix} 1.00 & 0.9240 & 0.3618 & 0.1139 \\ 0.9364 & 1.00 & 0.1991 & 0.0326 \\ 0.0000 & 0.0000 & 1.00 & 1.0103 \\ 0.3931 & 0.2364 & 0.9364 & 1.00 \end{matrix}\right\|. \quad (126)$$

$$Lmin = \lambda(X^0) = \left\|\begin{matrix} 0 & -0.0000 & 1.0000 & 1.0103 \\ 0.0000 & 0 & 1.0000 & 1.0103 \\ -0.0276 & 0.3574 & 0 & 0.0826 \\ 0.4660 & 0.7505 & 0.0756 & 0 \end{matrix}\right\|$$

The analysis of sizes of criteria in the relative estimates (120)-(125) shows that at the points of the optimum $X^* = \{X_1^*, X_2^*, X_3^*, X_4^*\}$ (on diagonal) the relative assessment is equal to unit.

Other criteria (102) $\lambda(X^*) = \|\lambda_q(X_k^*)\|_{q=\overline{1,K}}^{k=\overline{1,K}}$ there is much less unit.

It is required to find such point (parameters of the material) in the VPMP at which the relative estimates are closest to unit.

The solution of this problem is directed to the solution of λ-problem - step 4. 5.

Step 4. *Creation of the λ-problem is carried out in two stages*. Originally, the maximine problem of optimization with the normalized criteria is under construction:

$$\lambda^o = \boldsymbol{max_{X\in S} min_{k\in K}} \lambda_k(X), G(X)\leq 0, X \geq 0, \quad (127)$$

which at the second stage will be transformed to a reference problem of mathematical programming (λ-problem):

$$\lambda^o = max\,\lambda, \quad (128)$$

at restrictions $\lambda - \frac{f_1(X)-f_1^0}{f_1^*-f_1^0} \leq 0,\ \lambda - \frac{f_3(X)-f_3^0}{f_3^*-f_3^0} \leq 0, \lambda - \frac{f_2(X)-f_2^0}{f_2^*-f_2^0} \leq 0, \lambda - \frac{f_4(X)-f_4^0}{f_4^*-f_4^0} \leq 0,$

$0\leq\lambda\leq1,\ 20 \leq x_1 \leq 80; 4\leq x_2 \leq 60;, 2.10 \leq x_3 \leq 9.0, 2.2 \leq x_4 \leq 7.0.\ y_1 + y_2 + y_3 + y_4 = 102,$

where the vector of unknowns the λ-problem (128) has dimension of $N + 1$:

$\boldsymbol{X} = \{x_1, \ldots, x_N, \lambda\}, N = 4.$ By substituting the numerical values of the functions $f_1(X), f_2(X), f_3(X), f_4(X)$, we get λ-problem of the following form:

$$\lambda^o = max\,\lambda, \quad (129)$$

at restrictions $\lambda - \frac{323.84-2.249x_1-3.49*x_2 \ldots -0.2434x_3^2-0.5026x_4^2-f_1^0}{f_1^*-f_1^0} \leq 0,$

$\ldots \qquad \lambda - \frac{21-0.0097x_1-0.841x_2- \ldots +0.006x_3^2-0.0311x_4^2-f_4^0}{f_4^*-f_4^0} \leq 0,$

$0\leq\lambda\leq1,\ 21 \leq x_1 \leq 79;\ 5 \leq x_2 \leq 59;\ 2.10 \leq x_3 \leq 9;\ 2.2 \leq x_4 \leq 7.0.\ y_1 + y_2 + y_3 + y_4 = 102,$

Step 5. Solution of the λ-problem (129).

For this purpose we use the function *fmincon*(…), [16]:

$[Xo, Lo] = fmincon('Z\_Mater\_4Krit\_L', X0, Ao, bo, Aeq, beq, lbo, ubo,$
$'Z\_Mater\_LConst', options).$

As a result of the solution of VPMP of nonlinear programming (120)-(125) at equivalent criteria and λ-problem corresponding to it (129) received:

$$\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458, x_2 = 49.99, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}, \quad (130)$$

- an optimum point – design data of the technical system, $\boldsymbol{X^o}$, we will present in Figure 11.

$f_k(X^o), k = \overline{1, K}$ is sizes of characteristics of the technical system (criteria):

$$\{f_1(X^o) = 369.9, f_2(X^o) = 1759.6, f_3(X^o) = 194{,}9, f_4(X^o) = 49.3\}; \quad (131)$$

$\lambda_k(X^o), k = \overline{1, K}$ is sizes of the relative estimates

$$\lambda(X^o) = \{\lambda_1(X^o) = 0.7452, \lambda_2(X^o) = 0.6091, \lambda_3(X^o) = 0.6091, \lambda_4(X^o) = 0.6091\}; \quad (132)$$

$\lambda^o = 0.6091$ is the maximum lower level among all relative estimates measured in the relative units: $\lambda^o = min(\lambda_1(X^o), \lambda_2(X^o), \lambda_3(X^o), \lambda_4(X^o)) = 0.6091$.

$\lambda^o = 0.6091$ also call the guaranteed result in the relative units, i.e. $\lambda_k(X^o), k = \overline{1, K}$ and according to the characteristic of the technical system $f_k(X^o), k = \overline{1, K}$ it is impossible to improve, without worsening at the same time other characteristics.

Let's notice that according to the theorem 1, in X^o point criteria 2, 3, 4: $\lambda_1(X^o), \lambda_3(X^o), \lambda_4(X^o)$ are contradictory. This contradiction is defined by equality of

$\lambda^o = min\big(\lambda_1(X^o), \lambda_2(X^o), \lambda_3(X^o), \lambda_4(X^o)\big), \lambda^o = 0.6091,$

and other criteria inequality $\{\lambda_k(X^o)\} > \lambda^o$.

Theorem 1 serves as the basis for determining the correctness of the solution of a vector problem. As a rule, the equation for two criteria is fulfilled for the two criteria:

$\lambda^o = \lambda_q(X^o) = \lambda_p(X^o),\ q, p \in \boldsymbol{K}, X \in S$

(in our example, such criteria are 2, 3, 4), for other criteria is defined as inequality.

Stage 6. *Geometric interpretation of the results of the VPMP solution with 4 parameters and 4 criteria into a two-dimensional coordinate system (with 2 parameters) in relative units.*

For a geometric interpretation of the results of the VPMP solution with 4 parameters and 4 criteria, we will introduce changes in the two-dimensional coordinate system (with 2 parameters) in relative units. In VPMP (120)-(125) the parameters are x_1 and x_3 are considered as variables, parameters y_2 and y_4 are considered permanent. In accordance with the outcome of the decision of VPMP (120)-(125), with equivalent criteria:

$\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{x_1 = 45.4458, x_2 = 49.99, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}\}$ (130).

Let's assign a dimension to the constant parameters: $x_2 = 49.99, x_4 = 2.2$.

As a result, VZMP (120)-(125) became two-dimensional.

Results of the solution of VZMP (120)-(125) with two variables x_1 and x_3.

In the designation of the results of the solution, an additional letter "o" X1omax was introduced.

1. *According to the first criterion*, the coordinates of the point by the maximum are equal to:

X1omax = {x_1 =47.6032 x_2 = 49.9964 x_3 =8.0000 x_4 =2.2000}.

Values of the four criteria at the point X1omax: FX1omax = {$\boldsymbol{f_1(X1omax) = 407.9}$ $f_2(X1omax)$ = 1401.5 $f_3(X1omax)$ =190.5 $f_4(X1omax)$ =74.4.

Values are relative to the estimates of the criteria at the pointX1omax:

LX1omax = {$\boldsymbol{\lambda_1(X1omax)}$ = 1.1666 $\lambda_2(X1omax)$ = 0. 9406

$\lambda_3(X1omax)$ =0.5046 $\lambda_4(X1omax)$ =0. 0192.

Coordinates of the point according to the first criterion for the minimum:

X1omin ={33.9000 49.9964 2.100 2.2000}.

Values of the four criteria at the point X1omin:

FX1omin ={1.0e+03 * 0.3100 2.1799 0.1966 0.0260}.

Values are relative to the estimates of the criteria at the pointY1omin:

LX1omin =(**0.0806** 0.2200 0.6505 1.1562).

2. *According to the second criterion*, the coordinates of the point, functions and relative estimates for the maximum and minimum are equal:

X2omax ={56.6036 49.9964 8.0000 2.2000}.

FX2omax ={1.0e+03 * 0.4368 1.1630 0.1914 0.0858}.

LX2omax ={1.4871 **1.1615** 0.5205 -0.2501}.

X2omin ={33.9000 49.9964 2.1000 2.2000}.

FX2omin ={1.0e+03 * 0.3100 2.1799 0.1966 0.0260}.

LX2omin ={**0.0806** 0.2200 0.6505 1.1562}.

3. *According to the third criterion*, the coordinates of the point, functions and relative estimates for maximum and minimum:

X3omax ={33.9000 49.9964 2.1000 2.2000}.

FX3omax ={1.0e+03 * 0.3100 2.1799 0.1966 0.0260}.

LX3omax ={0.0806 0.2200 **0.6505** 1.1562}.

X3omin ={78.9999 49.9964 8.0000 2.2000}.

FX3omin ={518.7746 602.8613 192.6992 112.8235}.

LX3omin ={2.3963 1.6800 **0.5536** -0.8846}.

4. *According to the fourth criterion*, the coordinates of the point, functions and relative estimates for maximum and minimum are equal:

X4omax ={38.7000 49.9964 2.1000 2.2000}.

FX4omax ={1.0e+03 * 0.3250 2.1021 0.1972 0.0301}.

LX4omax ={0.2471 0.2920 0.6662 **1.0614**}.

X4omin ={63.4651 49.9964 8.0000 2.2000 23.1979}.

FX4omin ={460.4076 986.2753 191.8170   94.3384}.

LX4omin ={1.7488   1.3251   0.5316   **-0.4499**}.

Let us present the results of the solution as a whole with two variable parameters x_1 and x_3 (***two-dimensional problem***):

Xopt =[Xopt(1,:)={47.6032   44.1968   8.0000   2.2000}, $\lambda_1(X1omax)$ =**1.1666**;
Xopt(2,:)= {56.6036   35.1964   8.0000   2.2000}, $\lambda_2(X2omax)$ =**1.1615**;
Xopt(3,:)= {32.9000   60.0000   2.1000   7.0000}, $\lambda_3(X3omax) = \mathbf{0.6505}$;
Xopt(4,:)= {37.7000   60.0000   2.1000   2.2000}, $\lambda_4(X4omax) = \mathbf{1.0614}$;
Xo(1:4)= {44.7082   50.7315   4.3603   2.2000}, $\lambda(Xo) = \lambda^o = \mathbf{0.6081}$. (133)

Characteristics of the technical system in relative units $\lambda_1(Xo), \lambda_2(Xo), \lambda_3(Xo), \lambda_4(Xo)$ are shown in Fig. 12 in three-dimensional space x_1 x_3 and λ, where the third axis λ is the relative estimate.

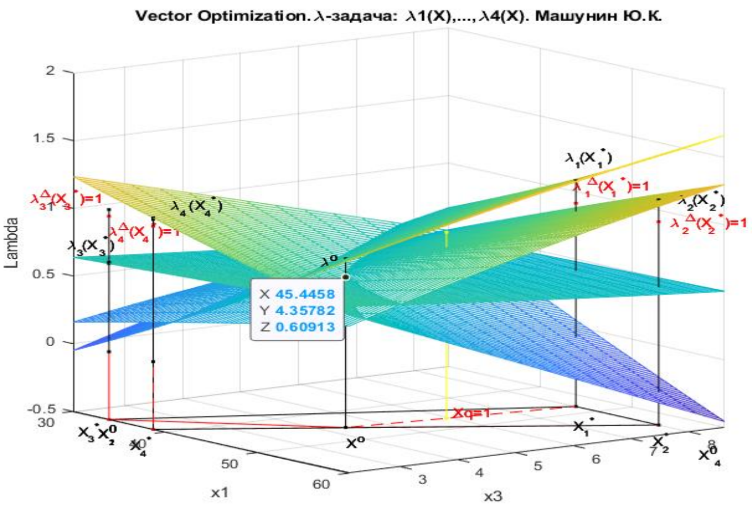

Figure 12. Geometric interpretation of four functions in relative units
$\lambda_1(Xo), \lambda_2(Xo), \lambda_3(Xo), \lambda_4(Xo)$in the three-dimensional coordinate system x_1 x_2 and λ.

In the admissible set of points ***S*** formed by constraints (124)-(125), the optimum points are $X_1^*, X_2^*, X_3^*, X_4^*$, combined into a contour, represent a set of Pareto-optimal points, $\boldsymbol{S^o \subset S}$ shown in Fig. 11. The coordinates of these points, as well as the characteristics of the technical system in relative units $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$are shown in Fig. 12 in three-dimensional space x_1 x_2 and λ, where the third axis λ is a relative estimate.

5.1.7. ***Stage 7****. Results of solving the Vector Problem of Mathematical Programming with equivalent criteria - preparation of information for decision-making with the priority of the criterion.*

As a result, the decision of the VPMP with equivalent criteria was received.

Results of solving a vector problem of mathematical programming.

(Mathematical model of the Engineering System (ES) ***with equivalent criteria***).

1. ***Criteria (characteristics of ES), parameters at the optimum point***:

Criteria* 1**: $f_1^* = f_1(X_1^*) = -392.9$; ***Parameters: $X_1^* = \{x_1 = 47.6, x_2 = 44.19, x_3 = 8., x_4 = 2.2\}$;
Criteria* 2**: $f_2^* = f_2(X_2^*) = 1337.38$; ***Parameters: $X_2^* = \{x_1 = 56.60, x_2 = 35.1964, x_3 = 8, x_4 = 2.2\}$;
Criteria* 3**: $f_3^* = f_3(X_3^*) = -210.57$; ***Parameters: $X_3^* = \{x_1 = 32.9000, x_2 = 59, x_3 = 2.1, x_4 = 7\}$;
Criteria* 4:** $f_4^* = f_4(X_4^*) = 32.68$; ***Parameters: $X_4^* = \{x_1 = 37.7, x_2 = 60, x_3 = 2.1, x_4 = 2.2\}$. (134)

2. ***Criteria (characteristics of ES), parameters at the antioptimum point***:

Criteria* 1**: $f_1^0 = f_1(X_1^0) = 302.7645$; ***Parameters: $X_1^0 = \{x_1 = 31.9, x_2 = 59.0, x_3 = 2.1, x_4 = 7.00\}$;
Criteria* 2**: $f_2^0 = f_2(X_2^0) = -2417.52$; ***Parameters: $X_2^0 = \{x_1 = 32.9, x_2 = 59.0, x_3 = 2.1, x_4 = 7.0\}$;

Criteria* 3**: $f_3^0 = f_3(X_3^0) = 170.52$; ***Parameters: $X_3^0 = \{x_1 = 78.99, x_2 = 10.4, x_3 = 8, x_4 = 4.59\}$;

Criteria* 4**: $f_4^0 = f_4(X_4^0) = -75.2$; ***Parameters: $X_4^0 = \{63.465, x_2 = 24.5, x_3 = 8, x_4 = 6\}$; (135)

3. ***Criteria (characteristics of ES), parameters at the optimum point*** in relative units at the point $\boldsymbol{X^o}$:

λ°=0.6091 – maximum lower level in relative units.

$\boldsymbol{\lambda_k}(\boldsymbol{X^o}), k = \overline{1, K}$- values of criteria (characteristics) in physical units:

$\{\lambda_1(X^o) = \mathbf{0.7452}, \lambda_2(X^o) = \mathbf{0.6091}, \lambda_3(X^o) = \mathbf{0.6091}, \lambda_4(X^o) = \mathbf{0.6091}\}$; (136)

Parameters: $\boldsymbol{X^o} = \{X^o = \{x_1 = 45.445, x_2 = 49.99, x_3 = 4.357, x_4 = 2.2, \lambda^o = 0.6091\}$. (137)

4. ***Criteria (characteristics of ES) in physical units***:

$\boldsymbol{f_k}(\boldsymbol{X^o}), \boldsymbol{k} = \overline{\boldsymbol{1}, \boldsymbol{K}}$- values of criteria (characteristics) in physical units:

$\{f_1(X^o) = 369.9, f_2(X^o) = 1759.6, f_3(X^o) = 194.9, f_4(X^o) = 49.3\}$; (138)

5. ***Engineering System Information in Physical Units for Criterion Priority Decision Making***:

1. $\boldsymbol{f_1(X) \rightarrow max}$: $f_1^* = f_1(X_1^*) = -392.9 \le f_1(X^o) = 369.9 \le f_1^0 = f_1(X_1^0) = 302.76$;

2. $\boldsymbol{f_2(X) \rightarrow min}$: $f_2^* = f_2(X_2^*) = 1337.38 \le f_2(X^o) = 1758.6 \le f_2^0 = f_2(X_2^0) = -2417.5$;

3. $\boldsymbol{f_3(X) \rightarrow max}$: $f_3^* = f_3(X_3^*) = -210.57 \le f_3(X^o) = 194.9 \le f_3^0 = f_3(X_3^0) = 170.53$;

4. $\boldsymbol{f_4(X) \rightarrow min}$: $f_4^* = f_4(X_4^*) = 32.68 \le f_4(X^o) = 49.3 \le f_4^0 = f_4(X_4^0) = -75.205$.

Conclusion on the Results of Solving *a Vector Problem of Mathematical Programming with Equivalent Criteria.*

We presented a methodology for solving a vector problem of mathematical programming with equivalent criteria (characteristics) on the example of an engineering system - the technical system. In the process of solving the problem, we presented the MATLAB system with a geometric interpretation of the solution results in a two-dimensional coordinate system. The results of solving a vector problem of mathematical programming with equivalent criteria with four variables coincide with the results of solving a vector problem with two variables.

Thus, for the first time in domestic and foreign practice, the transition and its geometric illustration from the *N*-dimensional to the two-dimensional dimension of the function in vector problems of mathematical programming with the corresponding errors of linear approximation are shown. We have prepared information for the study and selection of the optimal solution with the priority of one or another criterion (characteristic) of the engineering system. The next part of the work is aimed at solving this problem.

5.2. Part 2. Selection of optimal parameters of engineering systems under uncertainty - with the priority of the criterion based on multidimensional mathematics.

5.2.1. *Block 3. Analysis, comparison of the results of the VPMP solution with equivalent criteria of a complex engineering system (technical system model) and preparation for the selection of optimal parameters according to the priority criterion.*

Stage 8. *Analysis and comparison of the results of solving a vector problem with equivalent criteria with four and two variables*.

Let us compare the results of solving a vector problem of mathematical programming with equivalent criteria (120)-(125) with four variables x_1, x_2, x_3 and x_4 with the results of solving the vector problem (120)-(125) with two variables x_1 and x_3.

Results of solving a four-dimensional VPMP (120)-(125) with variable coordinates $\{x_1, x_2, x_3, x_4\}$ are presented in (130), (131), (132), and the results of solving a two-dimensional VPMP with variable coordinates $\{x_1\ x_3\}$ are presented in (133). The results of solving and comparing the studied problems are presented in a three-dimensional coordinate system $x_1\ x_3$. (Designation: $x_1\ x_3$ and λ in Fig. 13.

$\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{X^o = \{\boldsymbol{x_1} = \mathbf{45.4458}, x_2 = 49.9964, \boldsymbol{x_3} = \mathbf{4.3578}, x_4 = 2.2, \lambda^o = 0.6091\}$ of (130); and relative assessment:

$\{\lambda_1(Y^o) = 0.7452, \boldsymbol{\lambda_2(Y^o)} = \mathbf{0.6091}, \boldsymbol{\lambda_3(Y^o)} = \mathbf{0.6091}, \lambda_4(Y^o) = \mathbf{0.6091}\}$ from (131) in the same coordinates $x_1\ x_3$.

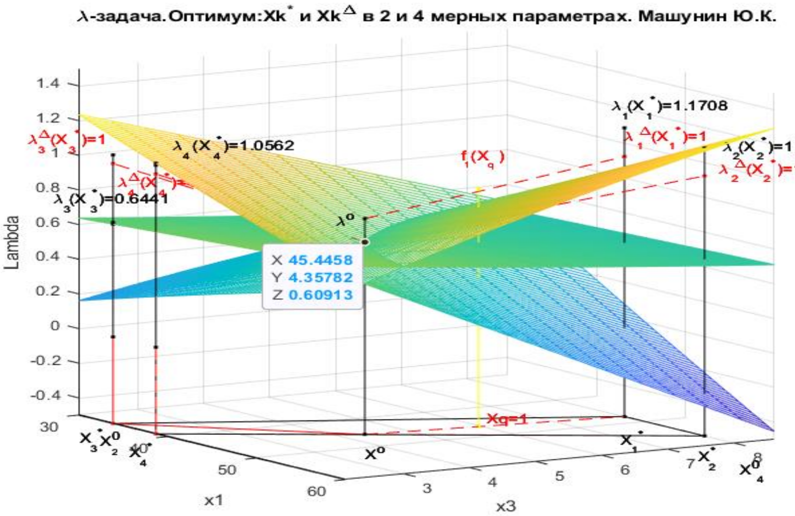

Figure 13. Solving λ-problems in a three-dimensional coordinate system $x_1\ x_3$ and λ

Consider, for example, the optimal point X_3^*. Function, $\lambda_3(X)$ is formed from the function $f_3(X)$ with variable coordinates $\{x_1\ x_3\}$, with constant coordinates $\{x_2$=49.54, x_4=2.2$\}$, taken from the optimal point X^o (130). At the point X_3^*. The relative estimate is $\boldsymbol{\lambda_3(X_3^*)} = \mathbf{0.6441}$ – shown in Fig. 13 with a black dot.

But we know that the relative estimate of $\lambda_3(X_3^*)$ obtained from the function $f_3(X_3^*)$. In the third step, it is equal to one, let's denote it as $\boldsymbol{\lambda_3^\Delta(X_3^*)} = \mathbf{1}$ – shown in Fig. 13 with a red dot.

In summary: *First*, with equivalent criteria for the coordinates of the optimum points x_1, x_3:

$\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{X^o = \{x_1 = \mathbf{45.44}, x_3 = \mathbf{4.3578}, \lambda^o = 0.6091\}$ - ***in a two-dimensional***

$\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{X^o = \{x_1 = \mathbf{45.44}, x_2 = 49.54, x_3 = \mathbf{4.3578}, x_4 = 2.2, \lambda^o = 0.6091\}$

in a four-dimensional system they coincide (using the software capabilities of the MATLAB system, we will show the numerical values of the optimum point $\{X^o, \lambda^o\}$ in Fig. 13);

secondly, at the optimum point X_3^* optimal values of the criteria $f_k(X_k^*), k = 3$ and relative estimates $\lambda_k(X_k^*) = 0.6441, k = 3$ and $\lambda_k^\Delta(X_k^*) = 1, k = 3$ do not match.

The difference between the relative estimates is $\lambda_3^\Delta(X_3^*)$=1 and $\lambda_3(X_3^*) = 0.6441$ is an error Δ=0.3559 in the transition from a four-dimensional (and in the general case *N*-dimensional) to a two-dimensional system of measurements. Similarly, in Fig. 13 shown:

point X_1^*, corresponding relative estimates $\lambda_1(X_1^*) = 1.1708$, $\lambda_1^\Delta(X_1^*) = 1$;
point X_2^*, corresponding relative estimates $\lambda_2(X_2^*) = 1.1530$, $\lambda_2^\Delta(X_2^*) = 1$;
point X_3^*, corresponding relative estimates $\lambda_3(X_3^*) = 0.6441$, $\lambda_3^\Delta(X_3^*) = 1$;
point X_4^*, corresponding relative estimates $\lambda_4(X_4^*) = 1.0562$, $\lambda_4^\Delta(X_4^*) = 1$.

Conclusion from the analysis: Information about the Engineering System in physical units for decision-making with the priority of the criterion.

Based on the results of solving a vector problem of mathematical programming with equivalent criteria, we formulated information for improving and making a management decision for any criterion (characteristics of an engineering system):

$f_k(X_k^*) \le \boldsymbol{f_k(X^o)} \le f_k(X_k^0), k = \overline{1, K}.$

Information about the Engineering System in physical units for decision-making:

1. $\boldsymbol{f_1(X) \to max}$: $f_1^* = f_1(X_1^*) = -392.9 \leq f_1(X^o) = 369.9 \leq f_1^0 = f_1(X_1^0) = 302.7645$;

2. $\boldsymbol{f_2(X) \to min}$: $f_2^* = f_2(X_2^*) = 1337.38 \leq f_2(X^o) = 1759.6 \leq f_2^0 = f_2(X_2^0) = -2452.3$;

3. $\boldsymbol{f_3(X) \to max}$: $f_3^* = f_3(X_3^*) = -210.57 \leq f_3(X^o) = 194.9 \leq f_3^0 = f_3(X_3^0) = 170.53$;

4. $\boldsymbol{f_4(X) \to min}$: $f_4^* = f_4(X_4^*) = 32.68 \leq f_4(X^o) = 49.3 \leq f_4^0 = f_4(X_4^0) = -75.205$.

5.2.2. Stage 9. Preparation of information for the selection of optimal parameters for any priority criterion and geometric interpretation of the solution results

When preparing information for the selection of optimal parameters according to the priority criterion and geometric interpretation of the results of solving a vector problem, we use Fig. 13, from which the functions $\{\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)\}$ in relative units, and represent it in Fig. 14.

Let's imagine in Fig. 14 Information data on relative estimates $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$ and physical data $f_1(X), f_2(X), f_3(X), f_4(X)$ obtained when solving a vector problem with equivalent criteria.

1. Let's present the results of the solution $\lambda_1(X_1^*), \dots, \lambda_4(X_4^*)$, which are obtained in the two-dimensional coordinate system x_1 x_3 and λ (see section 5.1.7):

point X_1^*, corresponding relative estimate $\lambda_1(X_1^*) = 1.1708$;

point X_2^*, corresponding relative estimate $\lambda_2(X_2^*) = 1.1530$;

point X_3^*, corresponding relative estimate $\lambda_3(X_3^*) = 0.6441$;

point X_4^*,, corresponding relative estimate $\lambda_4(X_4^*) = 1.0562$.

2. Theoretical results of the solution $\lambda_k^\Delta(X_k^*) = 1, k = \overline{1, K}, K = 4$ in the two-dimensional coordinate system x_1 x_3 and λ. (Red):

point X_1^*, corresponding relative estimate $\lambda_1^\Delta(X_1^*) = 1$;

point X_2^*, corresponding relative estimate $\lambda_2^\Delta(X_2^*) = 1$;

point X_3^*, corresponding relative estimate $\lambda_3^\Delta(X_3^*) = 1$;

point X_4^*, corresponding relative estimate$\lambda_4^\Delta(X_4^*) = 1$.

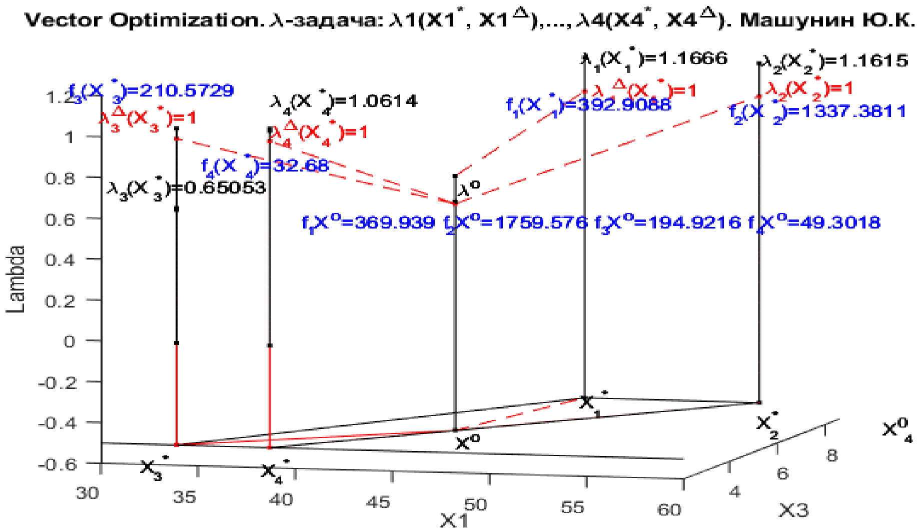

Figure 14. λ-problem in the coordinate system x_1 x_3 and λ. Geometric interpretation of the results of solving four criteria in relative and physical units.

1. Results of the solution $\lambda_1(X_1^*), \dots, \lambda_4(X_4^*)$ in a two-dimensional coordinate system x_1 x_3 and λ.
2. Theoretical results of the solution $\lambda_1^\Delta(X_1^*) = 1, \dots, \lambda_4^\Delta(X_4^*) = 1$ in a two-dimensional coordinate systemx_1 x_3 and λ.(Red color). ***3***. Results of solution $f_1(X_1^*) = \dots = f_4(X_4^*)$ in physical units, corresponding to $\lambda_1^\Delta(X_1^*) = 1, \dots, \lambda_4^\Delta(X_4^*) = 1$ – relative units. ***4***. Results of the solution $f_1(X_1^0), \dots, f_4(X_4^0)$ in physical units with equivalent criteria. (Purple).

3. Results of solution $f_1(X_1^*) = \cdots = f_4(X_4^*)$ in physical units, corresponding to $\lambda_1^\Delta(X_1^*) = 1, \dots, \lambda_4^\Delta(X_4^*) = 1$ – relative units that are obtained at the third step of the VPMP solution with equivalent criteria in (126) (Purple):

the point X_1^*, the corresponding physical quantity $f_1(X_1^*) = 392.908$;
point X_2^*, corresponding physical quantity $f_2(X_2^*) = 1337.38$;
point X_3^*, corresponding physical quantity $f_3(X_3^*) = 210.57$;
point X_4^*, corresponding physical quantity $f_4(X_4^*) = 32.68$.

4. Results of solving all criteria (characteristics of the technical system):

$f_1(X_1^0) = \cdots = f_4(X_4^0)$ in physical units at the optimum point $X_k^o, k = \overline{1,K}$ with equivalent criteria (Violet): $f_1 X_1^0 = 369.93, f_2 X_2^0 = 1759.57, f_3 X_3^0 = 194.92, f_4 X_4^0 = 49.3018$.

Let's analyze the results obtained. A linear function connecting the points $\lambda^o(X^o) = 0.6091$ and $\lambda_2^\Delta(X_2^*)$=1 in relative units characterizes the function $f_2(X)$ in relative units in the four-dimensional dimension of the parameters $x_1, \dots, x_4$. This linear function represents the geometric interpolation of the function $f_2(X)$ in relative units from *N*-dimensional (4-dimensional) to two-dimensional coordinate system. As can be seen from Fig. 14 Corresponding physical quantities $f_2(X)$ lie within the limits of:

$$f_2(X^o) = 1759.576 \le f_2(X) \le f_2(X_2^*) = 1337.38. \quad (139)$$

Note that the function $f_2(X)$ minimized – at the optimum point of X_2^*. The value is minimal. Similar (139) ratios are constructed in Fig. 14 for each criterion. Relations (139) are the basis for the study of the selected criterion in the sequence:

First, to select the value of the criterion in physical units:
secondly, the calculation of the corresponding relative valuation;
thirdly, the calculation of the relevant parameters of the technical system;
fourth, making a final decision.

The paper shows the choice of optimal parameters of the technical system for a given value of each criterion $k = \overline{1,K}, K = 4$.

The value of 1 of the criteria is given. $f_1(X) = 380$, which lies) within:
$f_1(X^o) = 369.9 \ge \boldsymbol{f_1(X) = 380} \ge f_1(X_1^*) = -392.9$. (***Block 4.k1***)
The value of criterion 2 is $f_2(X)$ = 1500, which lies) within:
$f_2(X^o) = -1759.6 \le \boldsymbol{f_2(X) = 1500} \le f_2(X_2^*) = -1337.38$. (***Block 4.k2***)
The value of 3 of the criteria is $f_3(X) = 200$, which lies) within:
$f_3(X^o) = 194.9 \le \boldsymbol{f_3(X) = 200} \le f_3(X_1^*) = -210.57$. (***Block 4.k3***)
Задана величина 4 критерия $f_4(X_1^0) = 40$, которая лежит) в пределах:

$$f_4(X^o) = -49.3 \ge \boldsymbol{f_4(X) = 40} \ge f_4(X_1^*) = -32.68. \text{ (Block 4.k4).} \quad (140)$$

The implementation of the selection of optimal parameters of the technical system for a given value of each criterion $\boldsymbol{f_k(X)}, k = \overline{1,K}, K = 4$ is presented in ***block 4***.

5.3. *Block 4. Study and selection of optimal parameters of the technical system according to priority criteria k1, k2, k3, k4 and geometric interpretation of the solution results in relative and physical units based on multidimensional optimization.*

The selection of the optimal parameters of a complex engineering (technical) system and the geometric interpretation of the solution results according to the priority criterion corresponds to the number of criteria K=4: Block ***k1***, Block ***k2***, Block ***k3***, Block ***k4***.

Name of the block.

Block ***k1***, ..., Block ***k4***. ***Solution of VPMP*** - a model of a complex engineering system with a given priority of the first criterion in multidimensional mathematics.

Each parameter selection block for the corresponding criterion includes the following sections.

1.k1, …, 1.k4. ***Solution of a vector problem*** - a model of a complex engineering system (technical system) with a given priority of the first criterion, including:

Solving a vector problem of mathematical programming - a model of an engineering system with equivalent criteria (Step 1).

Select the priority criterion $q \in \boldsymbol{K}$ (Step 2).

Select the numerical value of the selected priority criterion q(K (Steps 3, 4).

Solutions to a vector problem of mathematical programming - a model of an engineering system with a given priority of the $q \in \boldsymbol{K}$ criterion (Steps 5−8).

2.k1, …, 2.k4. Analysis of the results of solving a vector problem of mathematical programming - a model of an engineering system at a given priority of the first criterion.

3.k1, …, 3.k4. Geometric interpretation of the results of solving the vector problem of choosing optimal parameters by the priority criterion in relative units.

4.k1, …, 4.k4. Geometric interpretation of the results of the VPMP solution (functions: $f_1(X), \ldots, f_4(X)$) with the priority of the first criterion – the model of the technical system when designing in a three-dimensional coordinate system ***in physical units***.

1. Geometric interpretation of the results of the solution of **the first criterion** with the priority of the first criterion – the model of the engineering system when designing in a three-dimensional coordinate system ***in physical units. …***

4. Geometric interpretation of the results of the solution of the VPMP of **the fourth criterion** with the priority of the first criterion - the model of the engineering system when designing in a three-dimensional coordinate system ***in physical units***.

5.3.1. ***Block 4.k1. Selection of optimal parameters of the engineering system (technical system) according to the first priority criterion, geometric interpretation of the results of the VPMP solution.***

1.k1. Solution of VPMP - *a model of a complex engineering system with a given priority of the first criterion in multidimensional mathematics.*

The person making decisions, as a rule, is the designer of the technical system.

Step 1*. The solution of a VPMP with equivalent criteria.*

Results of the decision are presented in Stage 4.

Pareto-optimal points $\boldsymbol{S^o} \subset \boldsymbol{S}$ is between the optimal points $X_1^*\ X_{13}^o\ X_3^*\ X_{34}^o\ X_4^*\ X_{42}^o X_2^*\ X_{21}^o\ X_1^*$. Let us analyze the set of Pareto points $\boldsymbol{S^o} \subset \boldsymbol{S}$. For this purpose, we will connect the auxiliary points: $X_{13}^o\ X_{34}^o\ X_{42}^o\ X_{21}^o$ with the point X^o, which conventionally represents the center of the Pareto set.

As a result, the solution obtained four subsets of points $X \in \boldsymbol{S_q^o} \subset \boldsymbol{S^o} \subset \boldsymbol{S}, q = \overline{1,4}$.. A subset of points точек $\boldsymbol{S_1^o} \subset \boldsymbol{S^o} \subset \boldsymbol{S}$ – points $X_1^*\ X_{13}^o\ X^o X_{12}^o\ X_1^*$ is characterized by the fact that the relative score $\lambda_1 \geq \lambda_2, \lambda_3, \lambda_4$, i.e., in the field of the first criterion $\boldsymbol{S_1^o}$ takes precedence over the others. Similarly, $\boldsymbol{S_2^o}, \boldsymbol{S_3^o}, \boldsymbol{S_4^o}$ - subsets of points, is characterized by the fact that the second, third, and fourth criteria take precedence over the others, respectively.

The set of Pareto-optimal points is denoted by $\boldsymbol{S^o} = \boldsymbol{S_1^o} \cup \boldsymbol{S_2^o} \cup \boldsymbol{S_3^o} \cup \boldsymbol{S_4^o}$. The geometrical interpretation of the coordinates of all the obtained points and the relative estimates is presented in two-dimensional space $\{x_1\ x_3\}$ in Figure 11. These coordinates are shown in three measured spaces $\{x_1\ x_3\ \lambda\}$ in Figure 12, where the third axis λ is the relative valuation.

Limitations in Figure 12 lowered to -0.5 (to make the restrictions visible). The obtained information is the basis for further study of the structure of the Pareto set.

If results of the solution of a VPMP with equivalent criteria do not satisfy the person making the decision, then the choice of an optimal solution is carried out from any subset of points of $\boldsymbol{S_1^o}, \boldsymbol{S_2^o}, \boldsymbol{S_3^o}, \boldsymbol{S_4^o}$.

These subsets of Pareto points are shown in Figure 12 as functions $f_1(X), f_2(X), f_3(X), f_4(X)$.

Step 2*. Choice of priority criterion of $q \in \boldsymbol{K}$.*

From the theory it is known that in an optimum point of X^o there are always two most contradictory criteria: $q \in \boldsymbol{K}$ and $v \in \boldsymbol{K}$ for which in the relative unit's precise equality is carried out:

$\lambda^o = \lambda_q(X^o) = \lambda_p(X^o), q, p \in \boldsymbol{K}, X \in S,$

and for the others it is carried out inequalities: $\lambda^o \leq \lambda_k(X^o), \forall k \in \boldsymbol{K},\ q \neq p \neq k.$

In the technical system model (120)-(125) and the corresponding λ-problem (129), these criteria are the second and third: $\lambda^o = \lambda_2(X^o) = \lambda_3(X^o)$= 0.6091.

As a rule, from a pair of conflicting criteria $\lambda^o = \lambda_1(X^o) = \lambda_3(X^o) =\ 0.6091$, a criterion chosen by the decision maker would be improved. Such criterion is called "priority criterion", we will designate it $q = 1 \in \boldsymbol{K}$.

This criterion is investigated in interaction with the first criterion of $q = 1 \in \boldsymbol{K}$.

Selection of a priority criterion: The display provides general information for decision-making: Criteria at the optimum point X^o:

$$\boldsymbol{f_k(X^o)} = \{f_1(X^o) = 369.9, f_2(X^o) = 1759.6, f_3(X^o) =\ 194.9, f_4(X^o) =\ 49.3\}. \quad (141)$$

Relative estimates in X^o:

$$\boldsymbol{\lambda_k(X^o)} = \{\lambda_1(X^o) = \mathbf{0.7452}, \lambda_2(X^o) =\ \mathbf{0.6091}, \lambda_3(X^o) = \mathbf{0.6091}, \lambda_4(X^o) = \mathbf{0.6091}\}. \quad (142)$$

Antioptimum:

$f_1^0 = f_1(X_1^0) =\ 302.7645;\ \ f_2^0 =\ f_2(X_2^0) = -2417.52;\ \ f_3^0 = f_3(X_3^0) = 170.53;\ \ f_4^0 = f_4(X_4^0) = -75.2.$

Conclusion: Criteria 2, 3, 4 are the most contradictory, of which we choose the priority one.

This section examines ***the first criterion*** (function) $q = 1 \in \boldsymbol{K}$**.**

On the display the message is given:

q=input ('Enter priority criterion (number) of q =') - Entered: *q*=1.

Step 3. Numerical limits of change in the value of the priority of the criterion $q = 1 \in \boldsymbol{K}$ are formed. For the priority criterion $q = 1 \in \boldsymbol{K}$ changes in the numerical limits in natural units are determined during the transition from the point of the optimum X^o (116) to the point X_q^*, obtained in the first step. Information on criterion *q*=1 is displayed on:

$$f_q(X^o) =\ 369.9 \leq f_q(X) \leq 392.9 =\ f_q(X_q^*), q = 1 \in \boldsymbol{K}. \quad (143)$$

In the relative units the criterion of *q*=1 changes in the following limits:

$\lambda_q(X^o) = 0.7452 \leq \lambda_q(X) \leq 1 = \lambda_q\left(X_q^*\right), q = 1 \in \boldsymbol{K}.$

These data it is analysed.

Step 4. *Choice of size of priority criterion* of $q \in \boldsymbol{K}$. (Decision-making).

On the message: "Enter the size of priority criterion *fq*=" - we enter, the size of the characteristic defining structure of material: $\boldsymbol{f_q = 380}$.

Step 5. *The relative assessment is calculated*.

For the chosen size of priority criterion $f_q = 380$ the relative assessment is calculated:

$\lambda_q = \frac{f_q - f_q^0}{f_q^* - f_q^0} = \frac{380 - 369.93}{392.9 - 369.93} =\ 0.8568,$

which upon transition from X^o point to X_3^* lies in limits:

$$\lambda_q(X^o) =\ 0.7372 \leq \lambda_q(X) = 0.8568 \leq 1.0 =\ \lambda_q(X_q^*), q = 1 \in \boldsymbol{K}. \quad (144)$$

Step 6. Let's calculate coefficient of the linear approximation

Assuming the linear nature of change of criterion of $f_q(X)$ in (119) and according to the relative assessment of λ_q in (120) , using reference methods of the linear approximation, we will calculate a constant of proportionality between $\lambda_q(X^o)$, $\lambda_q$which we will call ρ:

$$\rho\ = \frac{\lambda_q - \lambda_q(X^o)}{\lambda_q(X_q^*) - \lambda_q(X^o)} = \frac{0.8568 - 0.6091}{1 - 0.6091} = 0.4380, q = 1 \in \boldsymbol{K}. \quad (145)$$

Step 7. Let's calculate coordinates of a priority of criteria with dimension of f_q

Assuming the linear nature of change of a vector of

$X^q = \{x_1\ \ x_2\ x_3\ \ x_4\}, q = 1$ we will determine point coordinates with dimension of

$f_q = 380$, the relative assessment (132):

$$x^{q=3}_{\lambda=0.74} = \{x_1 = X^o(1) + \rho(X^*_q(1) - X^o(1)),$$
$$x_2 = X^o(2) + \rho\left(X^*_q(2) - X^o(2)\right),$$
$$x_3 = X^o(3) + \rho\left(X^*_q(3) - X^o(3)\right),$$
$$x_4 = X^o(4) + \rho\left(X^*_q(4) - X^o(4)\right)\}, \quad (146)$$

where $\{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}$.

$X^*_1 = \{x_1 = 47.6032, x_2 = 44.1968, x_3 = 8.0, x_4 = 2.2\}$.

As result of the decision (162) we will receive X^q point with coordinates:

$$X^q = \{x_1 = 46.3908, x_2 = 47.4561, x_3 = 5.9531, x_4 = 2.2000\}. \quad (147)$$

Step 8. Calculation of ***the main indexes of a point of X^q***.

For the received X^q point, we will calculate:

all criteria in physical units $f_k(X^q) = \{f_k(X^q), k = \overline{1,K}\}$,

$$f_k(X^{q=1}) = \{f_1(X^q) = 381.5, f_2(X^q) = 1597.9, f_3(X^q) = 190.4, f_4(X^q) = 58.6\}; \quad (148)$$

all relative estimates of criteria:

$$\lambda^q = \{\lambda^q_k, k = \overline{1,K}\}, \ \lambda_k(X^q) = \frac{f_k(X^q) - f^0_k}{f^*_k - f^0_k}, k = \overline{1,K},$$

$$\lambda_k(X^q) = \{\lambda_1(X^q) = 0.8733, \lambda_2(X^q) = 0.7588, \lambda_3(X^q) = 0.4953, \lambda_4(X^q) = 0.3896\}; \quad (149)$$

Minimum relative estimates of criteria: $\min \lambda(X^q) = \min_{k \in K}(\lambda_k(X^q)) = 0.3896$.

$P^q = [p^1_1 = 1.0, p^1_2 = 1.1508, p^1_3 = 1.7632, p^1_4 = 2.2416]$;

vector priority criterion $P^q(X)$: $P^q(X) = \{p^q_k = \frac{\lambda_q(X^q)}{\lambda_k(X^q)}, k = \overline{1,K}\}$: $\lambda_k(X^q) * P^q = \{p^1_1 * \lambda_1(X^q) = 0.8733,\ p^1_2 * \lambda_2(X^q) = 0.8733,\ p^1_3 * \lambda_3(X^q) = 0.8733,\ p^1_4 * \lambda_4(X^q) = 0.8733\}$

Minimum relative estimates of criteria $q \in \boldsymbol{K}$:

$$\lambda^{oo} = \min(p^3_1\lambda_1(X^q), p^3_2\lambda_2(X^q), p^3_3\lambda_3(X^q), p^3_4\lambda_4(X^q)) = 0.8733$$

Other points from the Pareto region were obtained in a similar way $\boldsymbol{X^o_t} = \{\lambda^o_t, X^o_t\} \in \boldsymbol{S^o}$.

2.k1. Analysis of the Results of Solving a Vector Problem of Mathematical Programming - a Model of an Engineering System with a Given Priority of the First Criterion.

The calculated size of criterion $f_q(X^o_t)) = 381.7, q = 1 \in \boldsymbol{K}$ is usually not equal to the set $f_q = 380$. The error of the choice of $\Delta f_q = |f_q(X^o_t) - f_q| = |381.5 - 380| = 1.5$ is defined by an error of linear approximation, $\Delta f_{q\%} = 0.5\%$.

If error $\Delta f_q = |f_q(X^o_t) - f_q| = |381.5 - 380| = 1.7$, measured in physical units or as a percentage $\Delta f_{q\%} = \frac{\Delta f_q}{f_q} * 100 = 0.5\%$, is more than set $\Delta f, \Delta f_q > \Delta f$; we pass to a step 2, if $\Delta f_q \leq \Delta f$, calculations come to ***the end***.

In the course of modelling parametrical restrictions (101) and function (100) can be changed, i.e. some set of optimum decisions is received. Choose a final version which in our example included from this set of optimum decisions: parameters of the technical sistem:
$X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}$;

the parameters of the technicum system at a given priority criterion q=1:

$X^q = \{x_1 = 46.3908, x_2 = 47.4561, x_3 = 5.9531, x_4 = 2.2000\}$.

3.k1. Geometric Interpretation** of the Results of Solving the Vector Problem of Choosing Optimal Parameters by the First Priority Criterion **in Relative Units.

For a geometric interpretation of the results of solving the vector problem of choosing the optimal parameters for solving the vector problem according to the first priority criterion in relative units, we will form Fig. 15.*k1*.

Similarly, Fig. 14, we form relative estimates of the four criteria at the optimum point X_k^* $\lambda_k(X_k^*), k = \overline{1,K}$ (black) and $\lambda_k^\Delta(X_k^*) = 1, k = \overline{1,K}$ (red colour) and present them in Fig. 15.*k1*.

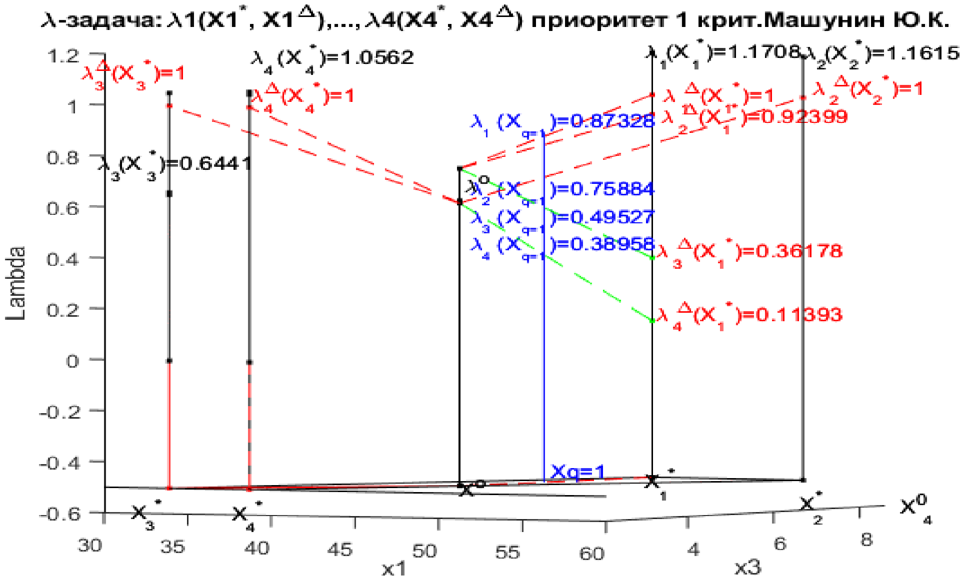

Figure 15. *k1*. λ-problem in the coordinate system x_1 x_3 and λ. Results of the solution: $\lambda_k^\Delta(X_k^*) = 1, k = \overline{1,K}$; relative estimates with the priority of the first criterion at the optimum point $X_{q=1}$: $\lambda_k(X_{q=1}), k = \overline{1,K}$.

At the third step, the solution of the VPMP with equivalent criteria at the optimum points X_1^*, X_2^*, X_3^*, X_4^* the values of all relative estimates are obtained:

$$L(X^*) = \left\| \begin{matrix} L(X_1^*) \\ L(X_2^*) \\ L(X_3^*) \\ L(X_4^*) \end{matrix} \right\| = \left\| \begin{matrix} \mathbf{1.00} & 0.9264 & 0.3618 & 0.1139 \\ 0.9364 & \mathbf{1.00} & 0.1991 & 0.0326 \\ 0.0000 & 0.0000 & \mathbf{1.00} & 1.0103 \\ 0.3931 & 0.2364 & 0.9364 & \mathbf{1.00} \end{matrix} \right\|.$$

Fig. 15.*k1* $\lambda_1(X_1^*) = 1.0000$ is denoted as $\lambda_2^\Delta(X_2^*)$=1.

Linear function connecting points $\lambda(X^o) = 0.7372$ and $\lambda_1^\Delta(X_1^*)$=1 in relative units characterizes the function $f_1(X)$ in relative units in the four-dimensional dimension of the parameters $x_1, \dots, x_4$. This linear function represents a geometric interpolation of the functions $f_1(X)$ in relative units from *N*-dimensional (4-dimensional) to two-dimensional coordinate system. All functions (criteria) are studied in the same way.

In general, Fig. 15.*k1* at X_1^* Geometric (linear) interpolation of all functions (criteria) is shown:

$f_1(X)$ and the corresponding relative assessment$\lambda_1^\Delta(X_1^*) = 1.0$;

$f_2(X)$ and the corresponding relative assessment $\lambda_2^\Delta(X_1^*) = 0.92451$;

$f_3(X)$ and the corresponding relative assessment $\lambda_3^\Delta(X_1^*) = 0.35608$;

$f_4(X)$ and the corresponding relative assessment $\lambda_4^\Delta(X_1^*) = 0.11216$.

Relative Estimates with the Priority of the First Criterion at the Optimum Point $X_{q=1}$ $\lambda_k(X_{q=1}) = \frac{f_k(X_{q=1}) - f_k^0}{f_k^* - f_k^0}, k = \overline{1,K}$: $\lambda_k(X_{q=1}) = \{\lambda_1(X_{q=1}) = 0.87913, \lambda_2(X_{q=1}) = 0.76544,$ $\lambda_3(X_{q=1}) = 0.48451, \lambda_4(X_{q=1}) = 0.3788\}$;

The results of the solution show that the mathematical results completely coincide with the geometric ones.

4.k1. Geometric interpretation** of the results of the VPMP solution (**functions: $f_1(X), \dots, f_4(X)$**) with the priority of **the first criterion** – the model of the technical system when designing in a three-dimensional coordinate system **in physical units.

The initial information for the geometric interpretation of the results of solving a vector problem (VPMP) with the priority of the first criterion is: parameters of the optimum point with equivalent criteria:

$\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458\,, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}$

calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (see Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing. Let us examine and present the parameters sequentially for each characteristic of the technical systems (criterion): $f_1(X), f_2(X), f_3(X), f_4(X)$ ***in physical units***.

***1. Geometric interpretation** of the results of the VPMP solution with the priority of the first criterion – **the first characteristic** $f_1(X)$ when designing in physical units.*

In Figure 16.*k1* investigated the optimum points X_1^*, $X_{q=1}$, with the corresponding relative estimates $\lambda_1(X_1^*)$ $\lambda_1(X_{q=1})$. Linear functions in coordinates were also investigated:

$\lambda^o - \lambda_1(X_{q=1})$ and $\lambda^o - \lambda_1^{\Delta}(X_1^*)$ in relative units, which characterize the function $f_1(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

The first characteristic of the technical systems $f_1(X)$ is formed in 4:

max $f_1(X) \equiv 323.84 - 2.25x_1 - 3.49x_2 + 10.72x_3 + 13.124x_4 + 0.0968x_1x_2 - 0.062x_1x_3 - 0.169x_1x_4 + 0.0743x_2y_3 - 0.1x_2x_4 - 0.0036x_3x_4 + 0.0143x_1^2 + 0.0118x_2^2 - 0.2434x_3^2 - 0.5026x_4^2$.

Let us present the geometric interpretation of the function $f_1(Y)$ in physical units with variable coordinates $\{x_1\ x_3\}$ and with constant coordinates $\{x_2 = 49.99, x_4 = 2.2\}$ in Fig. 16.*k1*.

When calculated ***by four variables***, $x_1, \dots, x_4$ coordinates of the point and function of the first criterion for four variables by ***maximum***: $X_1^* = \{x_1 = 47.6, x_2 = 44.1968, x_3 = 8.0, x_4 = 2.2\}$, $\boldsymbol{f_1^* = f_1(X_1^*) = -392.9}$, when calculated by four variables.

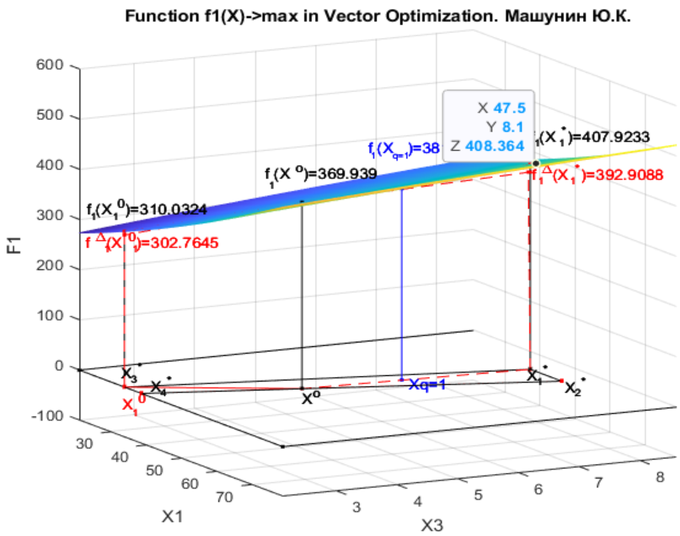

Figure 16.*k1*. Function $f_1(X)$ with the priority of ***the first criterion*** in the two-dimensional coordinate system $x_1\ x_3$ and a geometric interpretation of the function $f_1(X_{1,q=1})$ in the coordinate system $x_1\ x_2\ x_3\ x_4$.

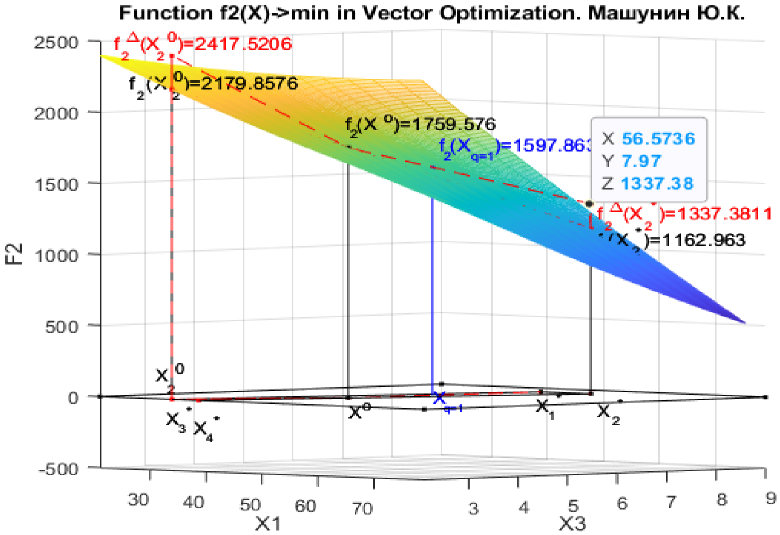

Figure 17.*k1*. The function $f_2(X)$ with the priority of ***the first criterion*** in the two-dimensional coordinate system x_1 x_3, geometric interpretation of the function $f_2(X_{1,q=1})$ are coordinates x_1 x_2 x_3 x_4 in physical units

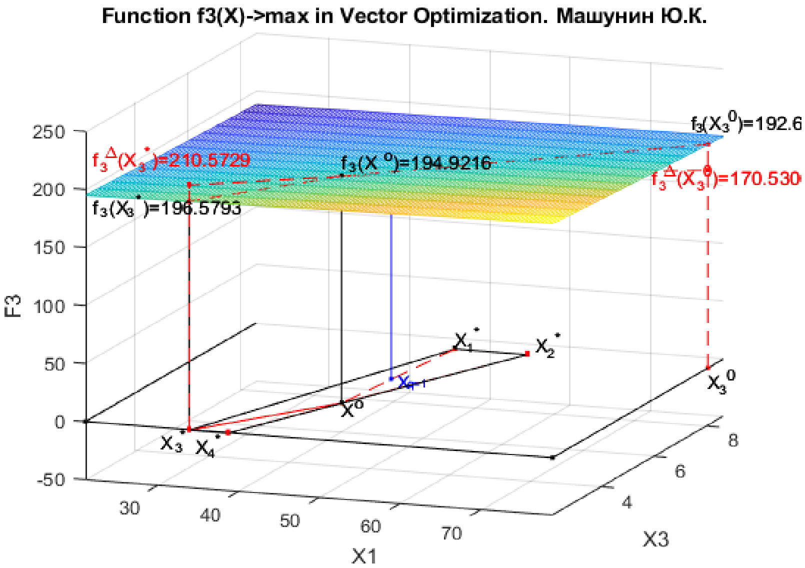

Figure 18.*k1*. Function $f_3(X)$ with the priority of ***the first criterion*** in the two-dimensional coordinate system x_1 x_3 and a geometric interpretation of the function $f_3(X_{1,q=1})$ in the coordinate system x_1 x_2 x_3 x_4.

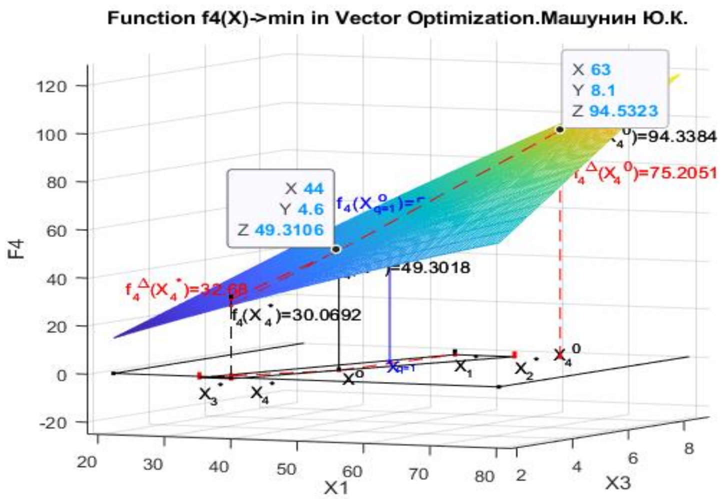

Figure 19.*k1*. Function $f_4(X)$ with the priority of ***the first criterion*** in the two-dimensional coordinate system x_1 x_3 and a geometric interpretation of the function $f_4(X)$ in the coordinate system x_1 x_2 x_3 x_4.

In the figure 16.*k1*, it is denoted as $f_1^{\Delta}(X_1^*)\ = -392.9$;

$X_1^*\ = \{x_1 = 46.5676, x_3 = 8.0\}$ (in Fig. 16.*k1* denoted as X_1^*). In a two-dimensional coordinate system y_1, y_3.

X1omax= $\{x_1\ = 47.6032, x_3\ = 8.0\}$ (Fig. 16.*k*1) is denoted as X_1^*) When calculated with two variables x_1, x_3 and two constants x_2, x_4.

The value of the objective function is equal to f_1^*=$f_1(X_1^*)$ = 407.92. (Black colour). In the MANLAB system, it is designated as h_1^*=Z=407.92.

The coordinates of the point and the function of the *first criterion to the minimum*:

At the point $X_1^0 = \{x_1\ = \mathbf{31.9},\ x_2\ = 59.0, \boldsymbol{x_3}\ = \mathbf{2.1}, x_4\ = 7.0\}$ value of the objective function f_1^0=$f_1(X_1^o)$ = 302.76. Value of the objective function in a four-dimensional coordinate system $f_1^{\Delta}(X_1^o)$ = 302.76. In the two-dimensional coordinate system $\boldsymbol{x_1}, \boldsymbol{x_3}$ the value of the objective function is equal to FX1omin =310.0. (Black color). Point coordinates X1omin=$\{x_1\ = 33.90, x_2\ = 49.9964,\ \ x_3\ = 2.10, x_4\ = 2.1\}$, value of the objective function FX1omin = 310.0.

Coordinates of the point and function of the first criterion *with equivalent criteria*:

The coordinates of the point are $X^o\ = \{x_1\ = 45.445, x_3\ = 4.3578\}$. Value of the objective function $f_1(X^o)$ = 369.9. $\lambda^o\ = \lambda_2(X^o)\ = \lambda_3(X^o)$= 0.6091.

Relative evaluation with priority of the first criterion at the optimum point $X_{q=1}$: $\lambda_1(X_{q=1}) = \frac{f_1(X_{q=1})-f_1^0}{f_1^*-f_1^0}$ = 0.87586.

In physical units, the value of the first criterion with priority at the optimum point is $X_{q=1}$ is equal to $f_1(X_{q=1})$ = 381.5 is close to the given $f_1(X_{q=1})$ = 380 (shown in purple).

A linear function connecting the points $f_1(X^o)$ and $f_1^{\Delta}(X_1^*)$ in physical units, it characterizes the function $f_1(X)$ in the four-dimensional dimension of the parameters $x_1, \ldots, x_4$.

And in general, the segments are $f_1^{\Delta}(X_1^*)$ -$f_1(X^o)$- $f_1^{\Delta}(X_1^0)$ represent the geometric interpretation of the function $f_1(X)$ in the four-dimensional dimension of the parameters $x_1, \ldots, x_4$.

2. Geometric interpretation of the results of the solution of the VPMP - the second characteristic $f_2(X_{2,q=1})$ *with the priority of* ***the first criterion*** *- the structure of the material in the design in physical units.*

Fig. 17.*k1* investigated: optimum points X_2^*, $X_{2,q=1}$, with corresponding relative estimates $\lambda_2(X_2^*)$ $\lambda_2(X_{2,q=1})$, as well as linear functions $\lambda^o\lambda_2(X_{2,q=1}) - \lambda_2^\Delta(X_2^*)$ in relative units, which characterize the function $f_2(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

The second characteristic of the structure of the technical system $f_2(X)$ (112) is formed in 4.1.4:

$min\ f_2(X)$≡795.72 +23.89x_1+30.866x_2 -25.8586x_3 -45.0026x_4 -0.7683x_1x_2 +0.4703x_1x_3 +0.7472x_1x_4-0.1283x_2x_3+0.3266x_2x_4-0.0112x_3x_4+0.0398x_1^2+0.0365x_2^2 +3.2x_3^2+2.6457x_4^2.

Let us present a geometric interpretation of the function $f_2(X)$ in physical units with variable coordinates $\{x_1\ x_3\}$ and with constant coordinates $\{x_2 = 49.54, x_4 = 2.2\}$, taken from the optimal point X^o (130) in Fig. 17.*k1*.

Coordinates of the point and the function of the second criterion on *the optimum* (*minimum*):

$f_2^* = f_2(X_2^*)\ = 1337.4.$ Parameters: $X_2^* = \{x_1 = 56.60, x_2 = 35.1964, x_3 = 8, x_4 = 2.2\}$ when calculated by four variables. In the figure, it is labelled as $f_2^\Delta(X_2^*)\ = 1337.4.$

In the MATLAB system, it is designated as $f_2^\Delta(X_2^*) = Z = 1337.38.$

$X_2^* = \{x_1 = 55.6, x_3 = 8.0\}$ (in Fig. 17.*k1* denoted as X_2^*).

In a two-dimensional coordinate system x_1, x_3. the value of the objective function is equal to f_2^*=$f_2(X_2^*) = 1163.06.$ (Black colour).

The coordinates of ***the minimum*** point (Worst Case - Maximum) at the point

$X_2^0 = \{x_1 = 31.9, x_3 = 2.1\}$ value of the objective function F_2^0=$f_2(X_2^o) = 2179.9.$

Value of the objective function in a four-dimensional coordinate system$f_2^\Delta(X_2^o) = 2417.5.$

The coordinates of the point and the function of ***the second criterion with equivalent criteria***: The coordinates of the point are $X^o = \{x_1 = 43.9, x_3 = 4.348\}$. The value of the objective function is $f_2(X^o) = 1759.6.$

Relative evaluation with priority of the first criterion at the optimum point $X_{q=1}$ of the second criterion: $\lambda_2(X_{q=1}) = \frac{f_2(X_{q=1}) - f_2^0}{f_2^* - f_2^0} = 0.76574.$

In physical units, the value of the first criterion with priority at the optimum point is $X_{q=1}$ is equal to $f_2(X_{q=1}) = 1597.9.$ (Shown in purple).

A linear function connecting the points $f_2(X^o)$ and $f_2^\Delta(X_2^*)$ in physical units, it characterizes the function $f_2(X)$ in the four-dimensional dimension of the parameters

$x_1, \dots, x_4$. And in general, the segments are $f_2^\Delta(X_2^*)$ -$f_2(X^o)$- $f_2^\Delta(X_2^0)$ represent the geometric interpolation of the function $f_2(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

3. Geometric interpretation of the results of the solution of the VPMP – the third characteristic $\boldsymbol{f_3}(X_{3,q=1})$ with the priority of **the first criterion** - the structure of the material in the design in physical units.

Figure 18.*k1* investigated the optimum points X_3^*, $X_{3,q=1}$, with the corresponding relative estimates $\lambda_3(X_3^*)$ $\lambda_3(X_{3,q=1})$. Linear functions in coordinates were also investigated:

$\lambda^o\lambda_3(X_{3,q=1}) - \lambda_3^\Delta(X_3^*)\,\lambda^o$ in relative units, which In characterize the function $f_3(X_{3,q=1})$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

The third characteristic of the technical system $f_3(X)$ is formed in 4:

$max\ f_3(X)$≡$110.22 + 0.7918x_1 + 1.73x_2 - 0.3713x_3 - 2.20x_4 - 0.0132x_1x_2 - 0.008x_1x_3 + 0.0193x_1x_4 - 0.0172x_2x_3 + 0.0161x_2x_4 - 0.0006x_3x_4 - 0.0004x_1^2 - 0.0002x_2^2 + 0.0335x_3^2 + 0.124x_4^2\}.$

Let us present the geometric interpretation of the function $f_3(X)$ in physical units with variable coordinates $\{x_1\ x_3\}$ and with constant coordinates $\{x_2 = 49.54, x = 2.2\}$ in Fig. 18.*k1*. The coordinates of the point and the function of *the third criterion to the maximum*:

$X_3^* = \{x_1 = 33.90, x_2 = 59.00, x_3 = 2.1, x_4 = 7.0\}, f_3^* = \ f_3(X_3^*)\ = -210.57,$

when calculated by four variables. In the figure 18.*k1*, it is denoted as $f_3^{\Delta}(X_3^*)\ = -210.57$
$X_3^* = \{x_1 = 33.9, x_3 = 2.1\}$ (in Fig. 18.*k1* denoted as X_3^*).

In a two-dimensional coordinate system x_1, x_3. The value of the objective function is equal to f_3^*=$f_3(X_3^*) = 196.83$. (Black colour). Value of the objective function $f_3^* = 196.83$.

The coordinates of the point and the function of the *third criterion to the minimum*:

At the point $X_3^0 = \{x_1 = 78.16, x_3 = 8.0\}$ value of the objective function

f_3^0=$f_3(X_3^o) = 192.69$. Value of the objective function in a four-dimensional coordinate system $f_3^{\Delta}(X_3^o) = 170.53$. Coordinates of the point and function of the third criterion *with equivalent criteria*: The coordinates of the point are $X^o = \{x_1 = 43.9, x_3 = 4.348\}$. Value of the objective function $f_3(X^o)$ =194.27. Relative evaluation with priority of the first criterion at the optimum point $X_{q=1}$ third criterion: $λ_3(X_{q=1}) = \frac{f_3(X_{3,q=1})-f_3^0}{f_3^*-f_3^0} = 0.48683$.

In physical units, the value of the third criterion with priority at the optimum point $X_{3,q=1}$ is close to the given $f_3(X_{q=1}) = 194.9$. (shown in purple).

A linear function connecting the points $f_3(X^o)$ and $f_3^{\Delta}(X_3^*)$ in physical units, it characterizes the function $f_3(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

And in general, the segments are$f_3^{\Delta}(X_1^*)$ -$f_3(X^o)$- $f_3^{\Delta}(X_3^0)$ represent the geometric interpretation of the function $f_3(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

4. Geometric interpretation of the results of the solution of the VPMP – the fourth characteristic $f_4(X_{4,q=1})$ with the priority of ***the first criterion*** of the material structure in the design ***in physical units.***

Figure 19.*k1* investigated the optimum points X_4^*, $X_{4,q=1}$, with the corresponding relative estimates $λ_4(X_4^*)$ $λ_4(X_{4,q=1})$. Linear functions in coordinates were also investigated:

$λ^o - λ_4(X_{4,q=1})$ and $λ^o - λ_4^{\Delta}(X_4^*)$ in relative units, which In characterize the function $f_4(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

The ***fourth*** characteristic of the structure of the material $f_4(X)$:

$max\ f_4(X)$≡$21.004 - 0.0097x_1 - 0.841x_2 - 0.4326x_3 + 1.1723x_4 + 0166x_1x_2 + 0.085x_1x_3 - 0.0001x_1x_4 + 0.0523x_2x_3\ + 0.0002x_2x_4 + 0.0006x_3x_4 - 0.0022x_1^2 + 0.0035x_2^2 + 0.006x_3^2 - 0.0311x_4^2\}\}$.

Let us present the geometric interpretation of the function $f_4(X_{4,q=1})$ in physical units with variable coordinates $\{x_1\ x_3\}$ and with constant coordinates $\{y_2 = 49.54, y_4 = 2.2\}$ in Fig. 18.*k1*.

The coordinates of the point and the function of *the fourth criterion to the maximum*:

$X_4^* = \{x_1 = 38.70, x_2 = 59.00, x_3 = 2.1, x_4 = 2.2\}, f_4^* = f_4(X_4^*)\ = 32.68$

when calculated by four variables. In the figure 18.*k1*, it is denoted as $f_4^{\Delta}(X_4^*)\ = 32.68$;

$X_4^* = \{x_1 = 38.70, x_3 = 2.1\}$. In a two-dimensional coordinate system x_1, x_3 the value of the objective function is equal to f_4^*=$f_4(X_4^*) = 30.1$. (Black colour). The value of the objective function is equal to F_4^*= 30.1.

The coordinates of the point and the function of the *fourth criterion to the minimum*:

At the point $X_4^0 = \{x_1 = 63.46, x_3 = 8\}$. Value of the objective function in a four-dimensional coordinate system $f_4^0 = f_4^{\Delta}(X_4^*)$=$-75.205$.

Coordinates of the point and function of the fourth criterion *with equivalent criteria*:

The coordinates of the point are $X^o = \{x_1 = 45.4458, x_3 = 4.3578\}$. Value of the objective function $f_4(X^o) = 49.3$.

Relative evaluation with priority of the first criterion at the optimum point $X_{q=1}$:

$λ_4(X_{q=1}) = \frac{f_4(X_{q=1})-f_4^0}{f_4^*-f_4^0} = 0.38154$.

In physical units, the value of the fourth criterion with priority at the optimum point is $X_{q=1}$ is equal to $f_4(X_{q=1}) = 58.6$ (shown in purple).

A linear function connecting the points $f_4(X^o)$ and $f_4^{\Delta}(X_4^*)$ in physical units, it characterizes the function $f_4(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

And in general, the segments are $f_4^{\Delta}(X_4^*)$ -$f_4(X^o)$- $f_4^{\Delta}(X_4^0)$ represent the geometric interpretation of the function $f_4(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

Conclusion on the section: criterion k1.

The section considers and solves the problem (fragment) of the development and adoption of a management decision under conditions of uncertainty in a complex engineering system (technical system). The analysis of the results of the solution of the VPMP at the given priority of ***the first criterion*** is carried out, a geometric interpretation of the results of the solution when designing four characteristics (criteria) in a three-dimensional coordinate system is presented, firstly, in relative units, and secondly, in physical units.

5.3.2. ***Block 4.k2. Selection of optimal parameters of the engineering system (technical system) according to the second priority criterion, geometric interpretation of the results of the VPMP solution, in relative and physical units***

The initial information for the geometric interpretation of the results of solving a vector problem with the priority of the second criterion is: parameters of the optimum point with equivalent criteria: $\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458\,, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}$ calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (see Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing.

The analysis of the results of the VPMP solution at the given priority of the second criterion is carried out and a geometric interpretation of the solution results when designing in a three-dimensional coordinate system ***in relative units*** is presented.

A geometric interpretation of the results of the VPMP solution with the priority of the second criterion when designing in a three-dimensional coordinate system ***in physical units*** is presented.

***1.k2. Solution of VPMP** - a model of a complex engineering system (**technical system) with a given priority of the second criterion in relative units** in multidimensional mathematics.*

The person making decisions, as a rule, is the designer of the technical system.

Step 1. *The solution of a VPMP with equivalent criteria*.

Results of the decision are presented in Stage 4.

Step 2. *Choice of priority criterion of $q \in \boldsymbol{K}$*.

From the pair $\lambda^o = \lambda_2(X^o) = \lambda_3(X^o)$=0.6091 contradictory criteria, the criterion that the decision-maker would like to improve is selected. The criterion is called the "priority criterion", $q = 2 \in \boldsymbol{K}$. It is studied in interaction with the criterion $q = 1 \in \boldsymbol{K}$.

Priority criterion selection: The display provides general information for decision-making: Criteria at the optimum point:

$X^o = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}$:

$$\boldsymbol{f_k}(\boldsymbol{X^o}) = \{f_1(X^o) = 369.9, \boldsymbol{f_2}(\boldsymbol{X^o}) = \mathbf{1759.6}, f_3(X^o) = 194.9, f_4(X^o) = 49.3\}. \quad (150)$$

Relative estimates in X^o: LXo ={0.7452 $\boldsymbol{\lambda_2}(\boldsymbol{X^o}) = \mathbf{0.6091}$ 0.6091 0.6091}. (151)

We examine these two criteria from the set of $\boldsymbol{K} = 4$ criteria shown in Fig. *15.k2*. A message is displayed on the display: q=input('Enter priority criterion (number) q= ') – Enter criterion ***q=2***.

Step 3. Numerical limits for changing the priority value of criterion q are formed:

$$f_2(X^o) = 1759.68 \leq f_2(X) \leq = 1337.38 = f_2(X_2^*), q \in \boldsymbol{K}. \quad (152)$$

Similarly, in relative units, the criterion q=2 varies within the following limits:

$\lambda_2(X^o) = 0.6091 \leq \lambda_2(X) \leq 1 = \lambda_2(X_2^*), q = 2 \in \boldsymbol{K}$. These data are analyzed.

Step 4. *Choice of size of priority criterion* of $q \in \boldsymbol{K}$. (Decision-making).

On the message: "Enter the size of priority criterion *fq*=" - we enter, the size of the characteristic defining structure of material: $\boldsymbol{f_q = 1500.}$

$f_2(X^o) = -1759.6 \leq \boldsymbol{f_2(X) = 1500} \leq f_2(X_2^*) = -1337.38$. Next, ***Step 5, …, Step 8***.

Geometric interpretation of the results of solving the vector problem of choosing optimal parameters according to the second priority criterion in relative units. Similarly, Fig. In Figure 15.*k*1, we form the relative estimates of the four criteria at the optimum point X_k^* $\lambda_k(X_k^*), k = \overline{1, K}$ (black) and $\lambda_k^\Delta(X_k^*) = 1, k = \overline{1, K}$ (red color). Let's imagine their Fig. 15.*k*2.

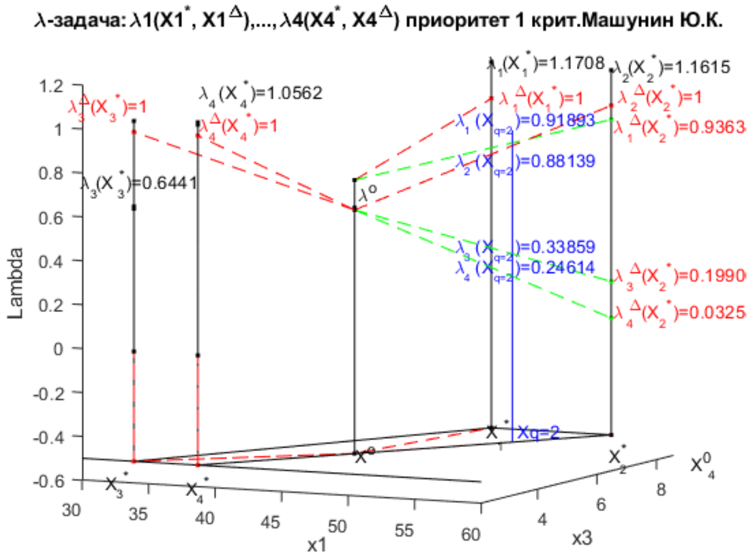

Figure 15.k2. λ-problem in the coordinate system x_1 x_3 and λ. Results of the solution: $\lambda_k^\Delta(X_k^*) = 1, k = \overline{1, K}$; relative estimates with the priority of the second criterion at the optimum point $X_{q=2}$: $\lambda_k(X_{q=2}), k = \overline{1, K}$.

Calculation of the main indexes of a point of X^q.
For the received X^q point, we will calculate:
all criteria in physical units $f_k(X^q) = \{f_k(X^q), k = \overline{1, K}\}$,
$f(X^q) = \{f_1(X^q) = 381.3,$

$$\boldsymbol{f_2(X^q) = 1465.7}, f_3(X^q) = 182.1, f_4(X^q) = 64.5\}; \quad (158)$$

all relative estimates of criteria: $\lambda^q = \{\lambda_k^q, k = \overline{1, K}\}$, $\lambda_k(X^q) = \frac{f_k(X^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}$,

$$\lambda_k(X^{q=2}) = \{\lambda_1(X^q) = 0.9189, \quad \lambda_2(X^q) = 0.8813, \quad \lambda_3(X^q) = 0.3385, \quad \lambda_4(X^q) = 0.2461\}; \quad (159)$$

2.k2. Analysis of results. The calculated size of criterion $f_q(X_t^o)) = 1465.7, q = 2 \in \boldsymbol{K}$ is usually not equal to the set $f_q = 1500$. The error of the choice of $\Delta f_q = |f_q(X_t^o) - f_q| = |1465.7 - 1500| = 34.3$ is defined by an error of linear approximation, $\Delta f_{q\%} = 0.5\%$.

If error $\Delta f_q = |f_q(X_t^o) - f_q| = |1465.7 - 1500| = 34.3$, measured in physical units or as a percentage $\Delta f_{q\%} = \frac{\Delta f_q}{f_q} * 100 = 0.5\%$, is more than set $\Delta f, \Delta f_q > \Delta f$; we pass to a step 2, if $\Delta f_q \leq \Delta f$, calculations come to ***the end***.

***4.k2. Geometric interpretation** of the results of the VPMP solution with the priority of the second criterion – the model of the technical system when designing in a three-dimensional coordinate system **in physical units**.*

The initial information for the geometric interpretation of the results of solving a vector problem (VPMP) with the priority of the first criterion is: parameters of the optimum point with equivalent criteria: $\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{Y^o = \{x_1 = 43.9, x_2 = 49.54, x_3 = 4.348, x_4 = 2.2\}, \lambda^o = 0.6087\}$, calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (see Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing.

By analogy with the drawings of Fig. 16.*k*1., ..., Fig. 19.*k*1, we will present:

Fig. 16.*k*2 the first criterion $f_1(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_1(X)$ with the priority of the second criterion $f_{q=2}=1500$;

Fig. 17.*k*2 the second criterion $f_2(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_2(X)$ with the priority of the second criterion $f_{q=2}=1500$;

Fig. 18.*k*2 the third criterion $f_3(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_3(X)$ with the priority of the second criterion $f_{q=2}=1500$;

Fig. 19.*k*2 the found criterion $f_4(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_4(X)$ with the priority of the second criterion $f_{q=2}=1500$.

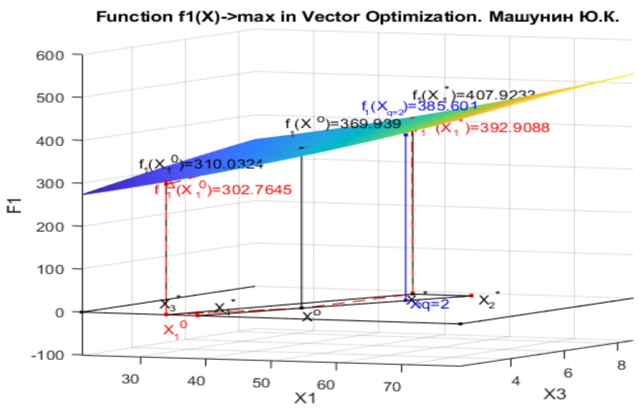

Figure 16.*k*2. Function $f_1(X)$ ***with the priority of the second*** criterion in the two-dimensional coordinate system x_1 x_3 and a geometric interpretation of the function $f_1(X_{2,q=2})$ in the coordinate system x_1 x_2 x_3 x_4.

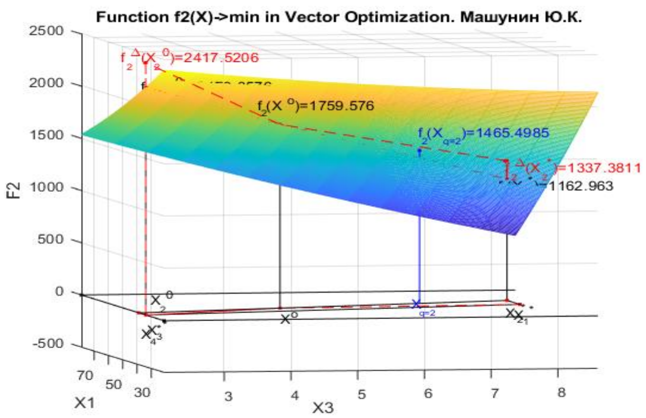

Figure 17.*k*2. The function $f_2(X)$ ***with the priority of the second*** criterion in the two-dimensional coordinate system x_1 x_3, geometric interpretation of the function $f_2(X_{2,q=2})$

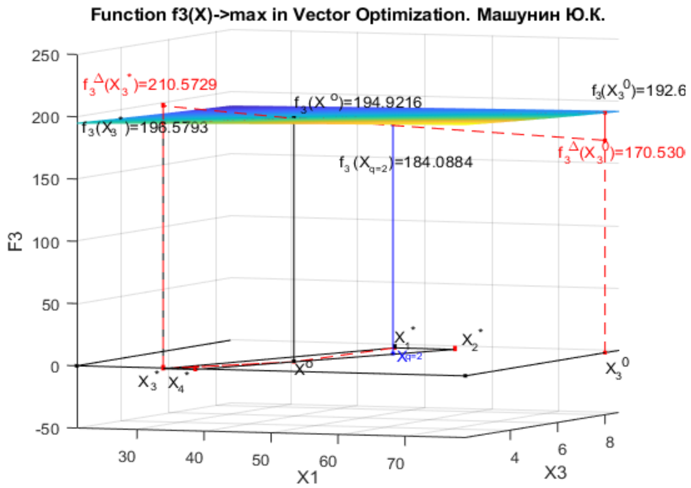

Figure 18.*k2*. Function $f_3(X)$ ***with the priority of the second criterion*** in the two-dimensional coordinate system $x_1\ x_3$ and a geometric interpretation of the function $f_3(X_{2,q=2})$ in the coordinate system $x_1\ x_2\ x_3\ x_4$.

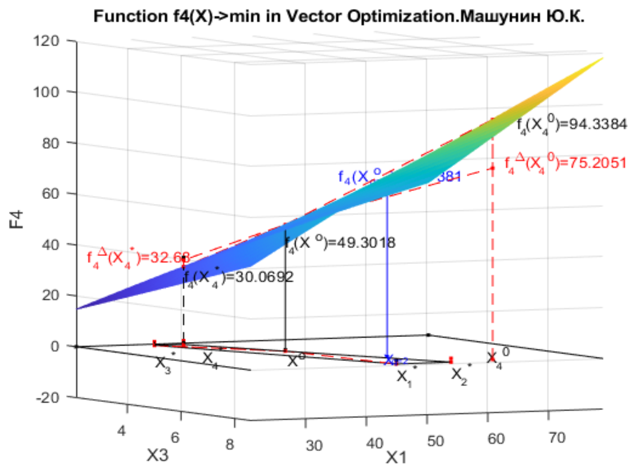

Figure 19.*k2*. Function $f_4(X)$ ***with the priority of the second criterion*** in the two-dimensional coordinate system $x_1\ x_3$ and a geometric interpretation of the function $f_4(X_{2,q=2})$ in the coordinate system $x_1\ x_2\ x_3\ x_4$.

5.3.3. ***Block 4.k3. Selection of optimal parameters of the engineering system (technical system) according to the third priority criterion, geometric interpretation of the results of the VPMP solution, in relative and physical units***

The initial information for the geometric interpretation of the results of solving a vector problem with the priority of the third criterion is: parameters of the optimum point with equivalent criteria: $\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458\,, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}$ calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (see Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing.

The analysis of the results of the VPMP solution at the given priority of the ***third*** criterion is carried out and a geometric interpretation of the solution results when designing in a three-dimensional coordinate system ***in relative units*** is presented.

A geometric interpretation of the results of the VPMP solution with the priority of the third criterion when designing in a three-dimensional coordinate system ***in physical units*** is presented.

***1.k3. Solution of VPMP** - a model of a complex engineering system (**technical system) with a given priority of the third criterion in relative units** in multidimensional mathematics.*

The person making decisions, as a rule, is the designer of the technical system.

***Step 1**. The solution of a VPMP with equivalent criteria.*

Results of the decision are presented in Stage 4.

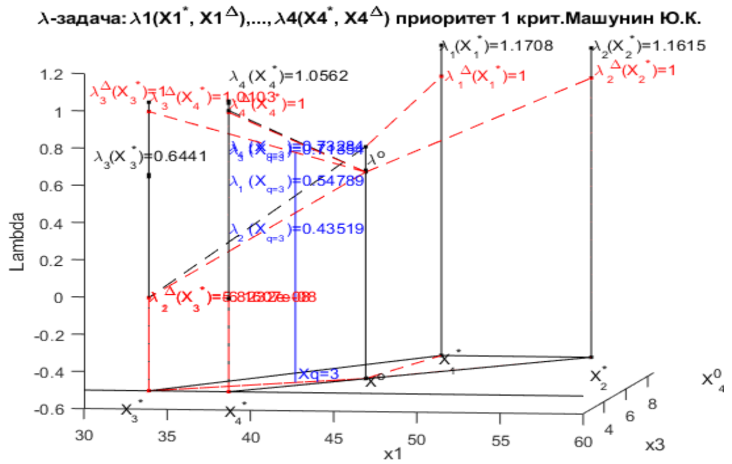

Figure 15.k3. λ-problem in the coordinate system x_1 x_3 and λ. Results of the solution: $\lambda_k^{\Delta}(X_k^*) = 1, k = \overline{1,K}$; relative estimates with the priority of the third criterion at the optimum point $X_{q=3}$:

$$\lambda_k(X_{q=3}), k = \overline{1,K}.$$

***Step 2**. Choice of priority criterion of $q \in \boldsymbol{K}$.*

From the pair $\lambda^o = \lambda_2(X^o) = \lambda_3(X^o)$=0.6091 contradictory criteria, the criterion that the decision-maker would like to improve is selected. The criterion is called the "priority criterion", $q = 3 \in \boldsymbol{K}$. It is studied in interaction with the criterion $q = 1 \in \boldsymbol{K}$.

Priority criterion selection: The display provides general information for decision-making: Criteria at the optimum point:

$$X^o = \{x_1 = 45.445, x_2 = 49.99,$$

$x_3 = 4.3, x_4 = 2.2, \lambda^o = 0.6091\}$:

$$\boldsymbol{f_k(X^o)} = \{f_1(X^o) = 369.9, f_2(X^o) = 1759.6, \boldsymbol{f_3(X^o)} = \mathbf{194.9}, f_4(X^o) = 49.3\}. \quad (153)$$

Relative estimates in X^o: LXo ={0.7452 0.6091 $\boldsymbol{\lambda_3(X^o)} = \mathbf{0.6091}$ 0.6091}. (154)

We examine these two criteria from the set of $\boldsymbol{K} = 4$ criteria shown in Fig. *15.k3*.

A message is displayed on the display: q=input('Enter priority criterion (number) q= ') – Enter criterion ***q=3***.

Step 3. Numerical limits for changing the priority value of criterion q are formed:

$$f_3(X^o) = 194.9 \leq \boldsymbol{f_3(X)} = \mathbf{200} \leq f_3(X_1^*) = -210.57. \quad (155)$$

Similarly, in relative units, the criterion q=1 varies within the following limits:

$\lambda_3(X^o) = 0.6091 \leq \lambda_3(X) \leq 1 = \lambda_3(X_2^*), q = 3 \in \boldsymbol{K}$. These data are analyzed.

Step 4**. Choice of size of priority criterion* of *q=4*∈K***. (Decision-making).

On the message: "Enter the size of priority criterion *fq*=" - we enter, the size of the characteristic defining structure of material: $\boldsymbol{f_{q=3}}$**=200.**

$$f_3(X^o) = 194.9 \leq \boldsymbol{f_3(X)} = \mathbf{200} \leq f_3(X_1^*) = -210.57. \quad (156)$$

Next, ***Step 5, …, Step 8***.

Geometric interpretation of the results of solving the vector problem of choosing optimal parameters according to the third priority criterion in relative units.

Similarly, Fig. In Figure 15.*k*1, we form the relative estimates of the four criteria at the optimum point X_k^* $\lambda_k(X_k^*), k = \overline{1,K}$ (black) and $\lambda_k^{\Delta}(X_k^*) = 1, k = \overline{1,K}$ (red color).

Solution results: relative estimates of four criteria with the priority of the third criterion at the optimum point $X_{q=3}$- $\lambda_k(X_{q=3}), k = \overline{1,K}$: $\lambda_1(X_{q=3}) = 0.7398, \dots, \lambda_4(X_{q=3}) = 0.4351$. (Purple). Let's imagine their Fig. 15.*k*3.

***4.k3. Geometric interpretation** of the results of the VPMP solution with the priority of the third criterion – the model of the technical system when designing in a three-dimensional coordinate system **in physical units**.*

The initial information for the geometric interpretation of the results of solving a vector problem (VPMP) with the priority of the third criterion is: parameters of the optimum point with equivalent criteria:

$$X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458\,, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\},$$

calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (see Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing.

By analogy with the drawings of Fig. 16.*k*1., ..., Fig. 19.*k*1, we will present:

Fig. 16.*k*3 the first criterion $f_1(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_1(X)$ with the priority of the third criterion $f_{q=3}$=*200*;

Fig. 17.*k*3 the second criterion $f_2(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_2(X)$ with the priority of the third criterion $f_{q=3}$=*200*;

Fig. 18.*k*3 the third criterion $f_3(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_3(X)$ with the priority of the third criterion $f_{q=3}$=*200*;

Fig. 19.*k*2 the found criterion $f_4(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_4(X)$ with the priority of the third criterion $f_{q=3}$=*200*.

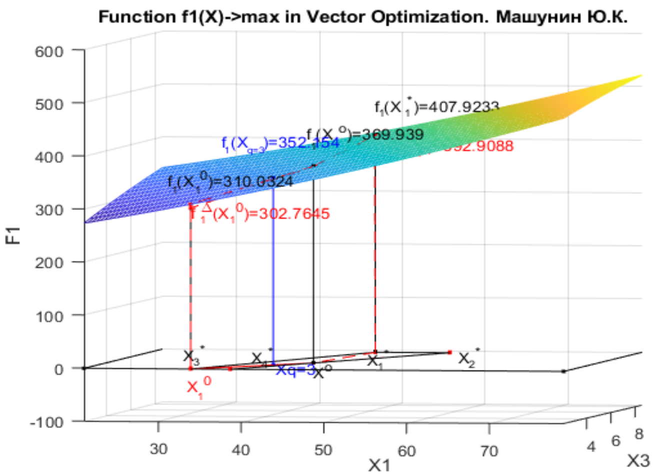

Figure 16.*k*3. Function $f_1(X)$ ***with the priority of the third*** criterion in the two-dimensional coordinate system x_1 x_3 and a geometric interpretation of the function $f_1(X_{3,q=3})$ in the coordinate system x_1 x_2 x_3 x_4.

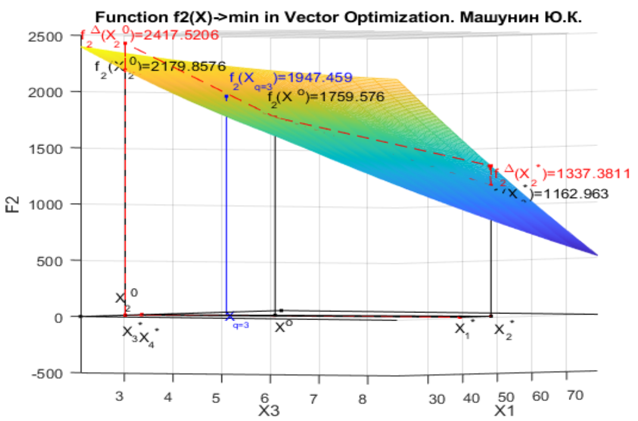

Figure 17.$k3$. The function $f_2(X)$ ***with the priority of the third*** criterion in the two-dimensional coordinate system x_1 x_3, geometric interpretation of the function $f_2(X_{3,q=3})$ are coordinates x_1 x_2 x_3 x_4.

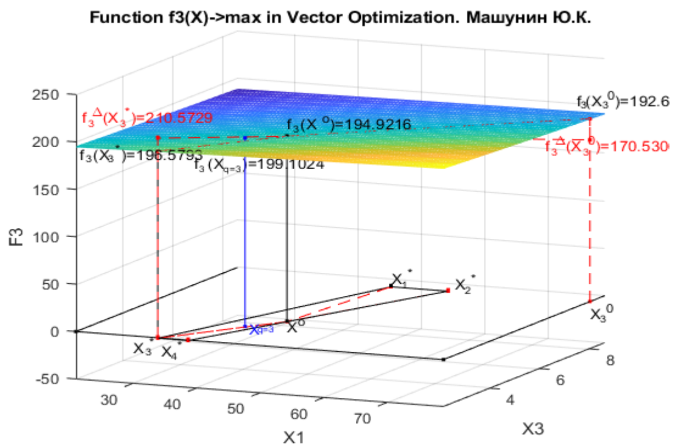

Figure 18.$k3$. Function $f_3(X)$ ***with the priority of the third criterion*** in the two-dimensional coordinate system x_1 x_3 and a geometric interpretation of the function $f_3(X_{3,q=3})$ in the coordinate system x_1 x_2 x_3 x_4.

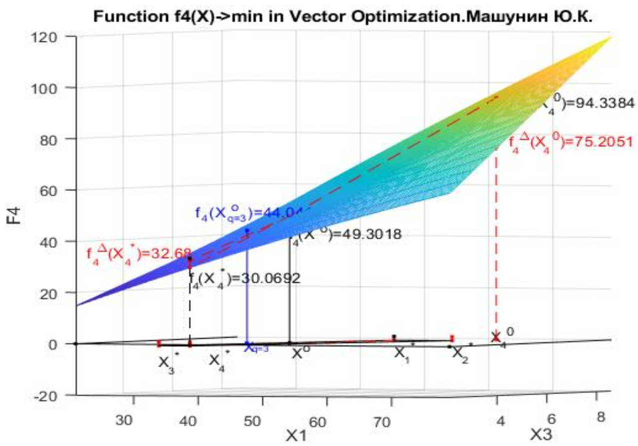

Figure 19.$k3$. Function $f_4(X)$ ***with the priority of the third criterion*** in the two-dimensional coordinate system x_1 x_3 and a geometric interpretation of the function $f_4(X_{3,q=3})$ in the coordinate system x_1 x_2 x_3 x_4.

Conclusion on the section: criterion k3.

The section considers and solves the problem (fragment) of the development and adoption of a management decision under conditions of uncertainty in a complex engineering system (technical system). The analysis of the results of the solution of the VPMP at the given priority of ***the third criterion*** is carried out, a geometric interpretation of the results of the solution when designing four characteristics (criteria) in a three-dimensional coordinate system is presented, firstly, in relative units, and secondly, in physical units.

5.3.4. *Block 4.k4. Selection of optimal parameters of the engineering system (technical system) according to the fourth priority criterion, geometric interpretation of the results of the VPMP solution, in relative and physical units.*

The initial information for the geometric interpretation of the results of solving a vector problem with the priority of the fourth criterion is: parameters of the optimum point with equivalent criteria: $\boldsymbol{X^o} = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458\,, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}$ calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (see Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing.

The analysis of the results of the VPMP solution at the given priority of the ***fourth*** criterion is carried out and a geometric interpretation of the solution results when designing in a three-dimensional coordinate system ***in relative units*** is presented.

A geometric interpretation of the results of the VPMP solution with the priority of the fourth criterion when designing in a three-dimensional coordinate system ***in physical units*** is presented.

***1.k4. Solution of VPMP** - a model of a complex engineering system (**technical system) with a given priority of the fourth criterion in relative units** in multidimensional mathematics.*

The person making decisions, as a rule, is the designer of the technical system.

***Step 1**. The solution of a VPMP with equivalent criteria.*

Results of the decision are presented in Stage 4.

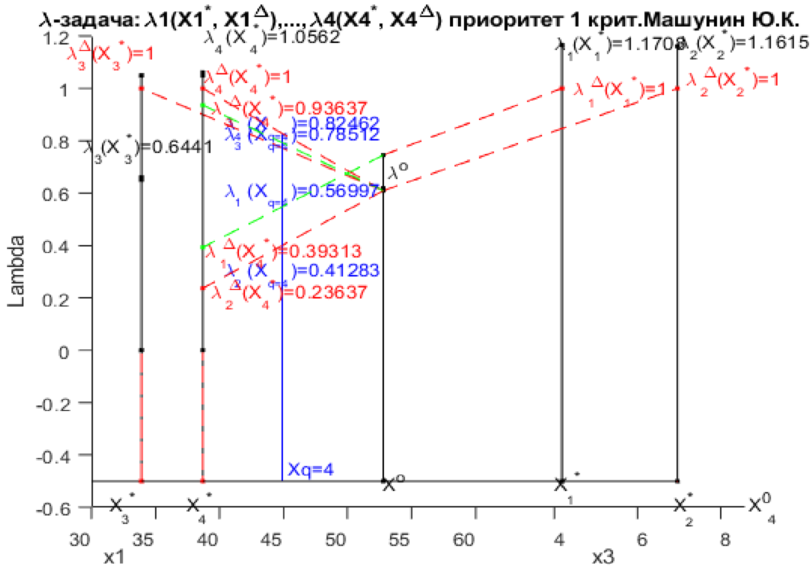

Figure 15.k4. λ-problem in the coordinate system $x_1\ x_3$ and λ. Results of the solution: $\lambda_k^{\Delta}(X_k^*) = 1, k = \overline{1,K}$; relative estimates with the priority of the fourth criterion at the optimum point $X_{q=4}$: $\lambda_k(X_{q=4}), k = \overline{1,K}$.

***Step 2**. Choice of priority criterion of* $q \in \boldsymbol{K}$.

From the pair $\lambda^o = \lambda_2(X^o) = \lambda_4(X^o)$=0.6091 contradictory criteria, the criterion that the decision-maker would like to improve is selected. The criterion is called the "priority criterion", $q = 4 \in \boldsymbol{K}$. It is studied in interaction with the criterion $q = 1 \in \boldsymbol{K}$.

Priority criterion selection: The display provides general information for decision-making: Criteria at the optimum point:

$$X^o = \{x_1 = 45.445, x_2 = 49.99,$$
$$x_3 = 4.3, x_4 = 2.2, \lambda^o = 0.6091\};$$
$$\boldsymbol{f_k(X^o)} = \{f_1(X^o) = 369.9,$$
$$f_2(X^o) = 1759.6,$$
$$f_3(X^o) = 194.9,$$
$$\boldsymbol{f_4(X^o)} = \boldsymbol{49.3}\}. \quad (157)$$

Relative estimates in X^o: LXo ={0.7452 0.6091 0.6091 $\boldsymbol{\lambda_3(X^o)} = \mathbf{0.6091}\}$. (158)

We examine these two criteria from the set of $\boldsymbol{K} = 4$ criteria shown in Fig. *15.k4*.

A message is displayed on the display: q=input('Enter priority criterion (number) *q*= ') – Enter criterion ***q=4***.

Step 3. Numerical limits for changing the priority value of criterion q are formed:

$$f_{q=4}(X^o) = 47.5045 \le f_{q=4}(X) \le 32.68 = f_q(X_q^*), q \in \boldsymbol{K}. \quad (159)$$

Similarly, in relative units, the criterion *q*=4 varies within the following limits:

$$\lambda_4(X^o) = 0.6091 \le \lambda_4(X) \le 1 = \lambda_4(X_4^*), q = 4 \in \boldsymbol{K}. \quad (160)$$

These data are analyzed.

Step 4**. Choice of size of priority criterion* of *q*=4∈K***. (Decision-making).

On the message: "Enter the size of priority criterion *fq*=" - we enter, the size of the characteristic defining structure of material: $\boldsymbol{f_{q=4}=40}$.

$f_4(X^o) = -49.3 \ge \boldsymbol{f_4(X)} = \mathbf{40} \ge f_4(X_1^*) = -32.68.$

Next, ***Step 5, …, Step 8***.

Geometric interpretation of the results of solving the vector problem of choosing optimal parameters according to the fourth priority criterion in relative units.

Similarly, Fig. In Figure 15.*k*4, we form the relative estimates of the four criteria at the optimum point X_k^* $\lambda_k(X_k^*), k = \overline{1,K}$ (black) and $\lambda_k^{\Delta}(X_k^*) = 1, k = \overline{1,K}$ (red color).

Solution results: relative estimates of four criteria with the priority of the third criterion at the optimum point $X_{q=4}$- $\lambda_k(X_{q=4}), k = \overline{1,K}$: $\lambda_1(X_{q=4}) = 0.8246, \dots, \lambda_4(X_{q=4}) = 0.4128$. (Purple). Let's imagine their Fig. 15.*k*4.

4.k4. Geometric interpretation** of the results of the VPMP solution with the priority of the fourth criterion* $f_{q=4}$*=40 – the model of the technical system when designing in a three-dimensional coordinate system **in physical units.

The initial information for the geometric interpretation of the results of solving a vector problem (VPMP) with the priority of the fourth criterion is: parameters of the optimum point with equivalent criteria:

$X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\},$
calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing.

By analogy with the drawings of Fig. 16.*k*1., ..., Fig. 19.*k*1, we will present:

Fig. 16.*k*4 the first criterion $f_1(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_1(X)$ with the priority of the *fourth* criterion $f_{q=4}$*=40*;

Fig. 17.*k*4 the second criterion $f_2(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_2(X)$ with the priority of the *fourth* criterion $f_{q=4}=40$;

Fig. 18.*k*4 the third criterion $f_3(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_3(X)$ with the priority of the *fourth* criterion $f_{q=4}=40$;

Fig. 19.*k*4 the found criterion $f_4(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_4(X)$ with the priority of the *fourth* criterion $f_{q=4}=40$.

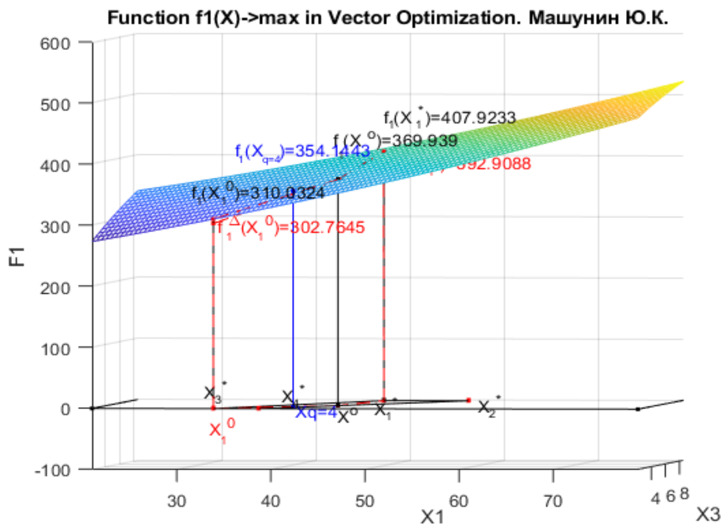

Figure 16.*k4*. Function $f_1(X)$ ***with the priority of the fourth*** criterion in the two-dimensional coordinate system $x_1\ x_3$ and a geometric interpretation of the function $f_1(X_{4,q=4})$ in the coordinate system $x_1\ x_2\ x_3\ x_4$.

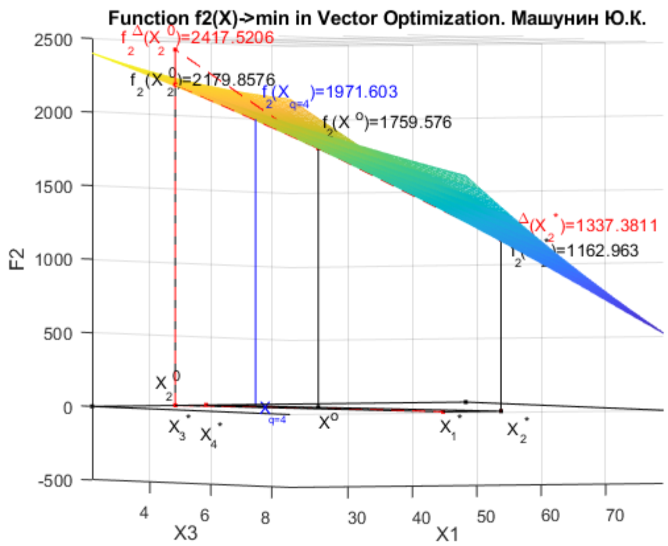

Figure 17.*k4*. The function $f_2(X)$ ***with the priority of the fourth*** criterion in the two-dimensional coordinate system $x_1\ x_3$, geometric interpretation of the function $f_2(X_{4,q=4})$ are coordinates $x_1\ x_2\ x_3\ x_4$.

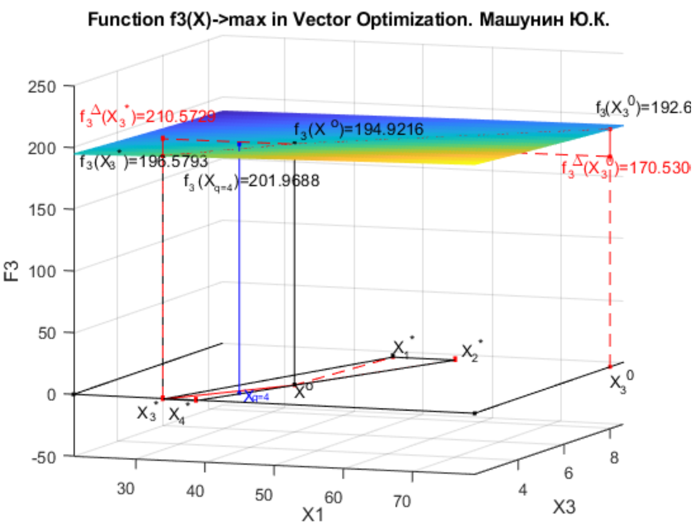

Figure 18.*k4*. Function $f_3(X)$ ***with the priority of the fourth criterion*** in the two-dimensional coordinate system x_1 x_3 and a geometric interpretation of the function $f_3\left(X_{4,q=4}\right)$ in the coordinate system x_1 x_2 x_3 x_4.

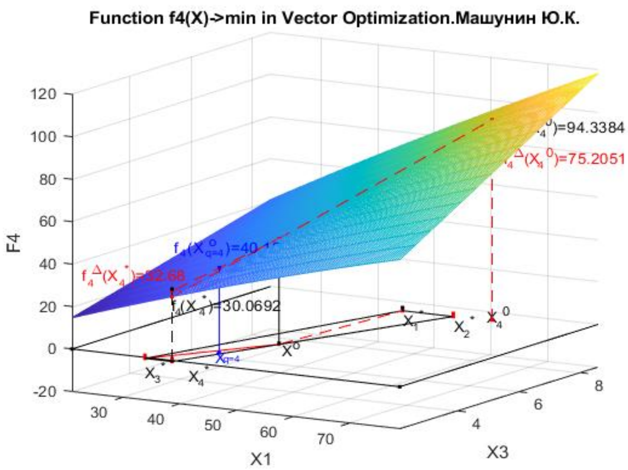

Figure 19.*k4*. Function $f_4(X)$ ***with the priority of the fourth criterion*** in the two-dimensional coordinate system x_1 x_3 and a geometric interpretation of the function $f_4\left(X_{4,q=4}\right)$ in the coordinate system x_1 x_2 x_3 x_4.

Conclusion on the section: criterion k4.

The section considers and solves the problem (fragment) of the development and adoption of a management decision under conditions of uncertainty in a complex engineering system (technical system). The analysis of the results of the solution of the VPMP at the given priority of ***the fourth criterion*** is carried out, a geometric interpretation of the results of the solution when designing four characteristics (criteria) in a three-dimensional coordinate system is presented, firstly, in relative units, and secondly, in physical units.

6. Let's evaluate mathematical results from the point of view of artificial intelligence.

Let us evaluate the results of this work from the point of view of artificial intelligence, obtained on the basis of the theory of vector optimization: {axiomatics of Y.K. Mashunin, principles of optimality and methods for solving vector problems of mathematical (convex) programming. Using the theory of vector optimization, we obtained, in particular, for the technical system:

Let's evaluate the applied methods of multidimensional mathematics - {axiomatics of Mashunin Yu.K., principles of optimality and methods for solving vector problems of mathematical (convex) programming with methods of artificial intelligence (AI).

1. A decision with equivalent criteria, which includes:

Optimum points are the parameters of the engineering system - $X^o = (x_j^o, j = \overline{1, N}\}$;

Characteristics (criteria) with equivalent criteria – $F(X^o) = \{f_k(X^o), k = \overline{1, K}\}$;

Relative evaluations with Equivalent Criteria – $\lambda(X^o) = \{\lambda_k(X^o), k = \overline{1, K}\}$, which lie within $\{0 \leq \lambda_k(X^o) \leq 1(\text{или } 100\%), k = \overline{1, K}\}$, and is easily converted into natural units.

2. A solution with a given priority of any criterion (characteristic), which includes:

Optimum points – parameters of an engineering system – with a given priority of any criterion (characteristic): X^q: $f_k(X^q) = \{f_1(X^q), f_2(X^q), f_3(X^q), f_4(X^q)\}, \forall k = \overline{1, K}$.

Formation of indicators (characteristics) at a given priority of any criterion at the optimum point X^q: $\lambda^q = \{\lambda_k^q, k = \overline{1, K},\ \lambda_k(X^q) = \frac{f_k(X^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}$.

Formation of relative estimates for a given priority of any criterion at the optimum point X^q: $\lambda^q = \{\lambda_k^q, k = \overline{1, K},\ \lambda_k(X^q) = \frac{f_k(X^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}$:

$\lambda_k(X^q) = \{\lambda_1(X^q), \lambda_2(X^q), \lambda_3(X^q), \lambda_4(X^q), \forall k = \overline{1, K}\}$.

Can these results be obtained by artificial intelligence, which, as a rule, functions on the principle of brute force? The answer is "No." Artificial intelligence can only get an approximate result, and this result must be evaluated by a human.

Thus, the developed theory of vector optimization can be a mathematical apparatus of computational intelligence of artificial intelligence.

7. Conclusions.

The problem of developing methods of multidimensional mathematics and making an optimal decision based on it in engineering research on a certain set of functional characteristics and experimental data is one of the most important tasks of system analysis and design.

The paper develops the theory and constructive methods for solving vector (multi-criteria) problems of mathematical programming, firstly, with equivalent criteria (characteristics of engineering systems), and secondly, with a given numerical value of the priority (of interest to the developer) criterion.

In the work, on the basis of vector optimization, a methodology for designing engineering systems has been developed by: 1. Building a mathematical model of an engineering system under conditions of certainty and uncertainty; development of constructive methods for solving a vector problem; 2. The construction of a numerical model for the selection of optimal parameters of a complex engineering system (many parametric and many functional) is presented; 3. The numerical implementation of the model of the technical system with equivalent criteria is presented; 4. A numerical implementation of the technical systems model is presented at a given priority of any criterion; 5. Presents a geometric interpretation of the results of the solution when designing four characteristics (criteria) in relative and physical units in a three-dimensional coordinate system.

The author is ready to participate in solving vector problems of linear and nonlinear programming.

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