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**ВЫБОР ОПТИМАЛЬНЫХ ПАРАМЕТРОВ В ЧЕТЫРЕХ ФУНКЦИОНАЛЬНОЙ
ТЕХНОЛОГИЧЕСКОЙ И ТЕХНИЧЕСКОЙ СИСТЕМЕ. МНОГОМЕРНАЯ
МАТЕМАТИКА И ИНТЕЛЛЕКТ КОНСТРУКТОРА
SELECTION OF OPTIMAL PARAMETERS IN THE FOUR FUNCTIONAL
TECHNOLOGICAL AND TECHNICAL SYSTEMS. MULTIDIMENSIONAL
MATHEMATICS AND CONSTRUCTOR INTELLIGENCE**

Abstract. Цель работы представить теорию, методы многомерной математика и интеллект конструктора в системе принятия оптимальных решений, в задачах возникающих при исследовании, проектировании и выборе параметров многофункциональных инженерных систем. В рамках теории векторной оптимизации представлены принципы оптимальности решения векторных задач при равнозначных критериях и при заданном приоритете критерия <https://rdcu.be/bhZ8i>. (Работа "Vector optimization with equivalent and priority criteria" Springer Nature распространяется бесплатно.). В многофункциональных математических моделях инженерных систем целенаправленность сформирована в виде улучшения множества характеристик, (критериев), таким образом, математические модели представлены задачами векторной оптимизации. В работе решение проблемы многофункциональности инженерных систем в выборе оптимальных параметров представлено в две стадии. На первой стадии работы показано решение выбора оптимальных параметров инженерной системы в условиях определенности и неопределенности при равнозначных критериях. На второй стадии работы, использованы интеллектуальные знания конструктора системы, который задает численные значения наиболее важного (приоритетного) критерия. Оптимальные значения параметров для приоритетного критерия вычисляется математически. Сформирована геометрическая интерпретация результатов решения N-мерной системы в двухмерную систему параметров при проектировании характеристик в относительных и физических единицах.

Abstract. The purpose of the work is to present the theory, methods of multidimensional mathematics and the intelligence of the designer in the system of optimal decision-making, in the problems arising in the research, design and selection of parameters of multifunctional engineering systems. Within the framework of the theory of vector optimization, the principles of optimality of solving vector problems with equivalent criteria and with a given priority of the criterion are presented <https://rdcu.be/bhZ8i> (The work "Vector optimization with equivalent and priority criteria" by Springer Nature is distributed free of charge.). In multifunctional mathematical models of engineering systems, purposefulness is formed in the form of improving a set of characteristics (criteria), thus, mathematical models are represented by vector optimization problems. In the work, the solution of the problem of multifunctionality of engineering systems in the selection of optimal parameters is presented in two stages. At the first stage of the work, the solution for choosing the optimal parameters of the engineering system under conditions of certainty and uncertainty with equivalent criteria is presented. At the second stage of work, the intellectual knowledge of the system designer was used,



who sets the numerical values of the most important (priority) criterion. The optimal parameter values for the priority criterion are calculated mathematically. A geometric interpretation of the results of the solution of the N -dimensional system into a two-dimensional system of parameters in the design of characteristics in relative and physical units is formed.

Ключевые слова: Многомерная математика, Методы принятия оптимальных решений, Моделирование инженерной системы, Интеллект конструктора, Принятие оптимальных решений, Геометрическая интерпретация многомерных систем.

Keywords: multidimensional mathematics, methods of optimal decision making, modeling of an engineering system, designer's intelligence, optimal decision making, geometric interpretation of multidimensional systems.

Introduction

Engineering systems include technical, technological systems and materials. The study of the development of engineering systems has shown that these systems depend on a certain set of functional characteristics, which together characterize the multidimensionality of the engineering system under study. The study and analysis of the functioning of engineering systems showed that the improvement in one of the characteristics of the system leads to the deterioration of other characteristics.

To improve the functioning of the engineering system, it is necessary to take into account: first, that in the engineering system under study, one subset of characteristics (criteria) was aimed at increasing the numerical value (maximization), and the second subset of the characteristics (criteria) of the system was aimed at decreasing the numerical value (minimization); Secondly, it is necessary that all the characteristics of the engineering system are improved in the aggregate. At present, there is a solution to one-criteria optimization, which can be interpreted as one-dimensional optimization.

The study of multi-criteria problems began more than a hundred years ago in the work of Pareto V. [1]. Over the past three decades, a large number of monographs and individual articles have been devoted to methods for solving vector (multi-criteria) problems. This is due to the widespread use of these methods in solving practical problems. The analysis of methods and algorithms for solving multi-criteria problems in accordance with its classification is presented in a number of works [6, 10, 22, 39, 45]: methods for solving multi-criteria problems based on the collapse of criteria with weight coefficients [3, 5, 25, 27]. The study of multi-criteria optimization was carried out both at the theoretical level by foreign [3, 25-30] and Russian authors [4-23], and at the solution of practical problems, first in the field of economics [31-45], and then in the field of engineering systems [6-21, 46-49]. The set of criteria in a multi-criteria optimization problem can be represented as a vector with which mathematical operations can be performed, hence the vector optimization problems or vector problems of mathematical programming.

Solving the problem of vector optimization is due to a number of difficulties, moreover, of a conceptual nature, and the main one is to understand: "what does it mean to solve the problem of vector optimization". To solve a problem with multiple criteria, it is mathematically necessary to create, first, axiomatics and principles of optimality, showing any user in what way one solution is better than another solution, and what is the optimal solution of a multi-criteria (characteristics) optimization problem. And at the next stage, on the basis of axiomatics and the principles of optimality, the development of constructive methods for solving multi-criteria problems.

The purpose of the work is to present the theory, methods of multidimensional mathematics and the intelligence of the designer in the system of optimal decision-making, in the problems arising in the research, design and selection of parameters in a multiparameter and multifunctional engineering system. The choice of optimal parameters is implemented with equivalent criteria and with a given priority of the criterion (characteristic). To present a geometric interpretation of the results of solving an N -dimensional system of parameters into a two-dimensional system of parameters.



Initially, the paper presents the theory and methods of multidimensional mathematics. Within the framework of the theory of vector optimization, axioms have been formulated and the principles of optimality of solving vector problems with equivalent criteria and with a given priority of the criterion have been presented [6–22]. The applied part presents constructive methods for making optimal decisions, methods for solving vector problems for modeling engineering systems. The methods allow you to make a decision, firstly, with equivalent criteria, and secondly, with a given priority of the criterion of a particular criterion. In the field of engineering systems, which include technical systems [11–14, 18, 19], technological processes [15, 21], materials [17].

On the basis of multidimensional mathematics, a methodology for modelling and designing engineering systems has been developed, which is implemented in two stages.

The first stage of work is aimed at solving the choice of optimal parameters of the engineering system in conditions of certainty and uncertainty with equivalent criteria and includes. 1. Construction of a mathematical model of an engineering system under conditions of certainty and uncertainty and the development of constructive methods for solving a vector problem. 2. Construction of a numerical model for the selection of optimal parameters of a complex engineering system. 3. A numerical implementation of a multiparameter and multifunctional model of an engineering system with equivalent criteria has been developed.

The second stage of work is aimed at solving the choice of optimal parameters of the engineering system in conditions of uncertainty, while using the intellectual knowledge of the system designer. 4. The designer of the technical system sets the numerical values of the most important (priority) criterion. 5. The optimal parameter values for the priority criterion are calculated mathematically. 6. A geometric interpretation of the results of solving an N -dimensional system of parameters into a two-dimensional system of parameters in the design of characteristics in relative and physical units is formed.

Mathematical and numerical implementation of the methodology for modeling and designing multiparameter and multifunctional engineering systems based on multidimensional mathematics using the intellectual designer is presented on two types of engineering systems. In the second chapter, we consider the technological system as the object of research. In the third chapter, we consider the technical system as the object of research.

1. Multidimensional Mathematics. Theory, methods.

In this chapter, we will present four sections characterizing the theory and methods of multivariate mathematics: Vector problem of mathematical programming;

Axiomatics, the principle of optimality and constructive methods of vector optimization with equivalent criteria;

Axiomatics, the principle of optimality and constructive methods of vector optimization with a given priority of the criterion;

Applied Multidimensional Mathematics: Vector Problem of Nonlinear Programming – A Model for the Development of Engineering Systems.

1.1. A vector problem of mathematical programming

1.1.1. Introduction to Multidimensional Mathematics.

As a representative of multidimensional mathematics, we will formulate the problem of mathematical programming, represented by a set of functions that determine the multidimensionality of the object under study. Each function of this set of functions has a different target orientation: maximization or minimization, which in the aggregate change on a certain (not empty and closed) set of variables (parameters).

Without breaking the generality, a set of functions can be represented in the form of a vector of functions. As a result, we get a vector problem of mathematical programming. On the basis of the vector optimization problem, we present the theoretical problems necessary for its solution, which include



axiomatics (theoretical foundations), the principle of optimality and constructive methods for solving vector problems with equivalent criteria and a given priority of the criterion, [6, 15, 29, 31, 33, 44].

1.1.2. A vector problem of mathematical programming

A vector problem in mathematical programming (VPMP) is a standard mathematical-programming problem including a set of criteria, which, in total, represent a vector of criteria.

It is important to distinguish between uniform and non-uniform VPMP:

A *uniform maximizing VPMP* is a vector problem in which each criterion is directed towards maximizing;

A *uniform minimizing VPMP* is a vector problem in which each criterion is directed towards minimizing;

A *non-uniform VPMP* is a vector problem in which the set of criteria is shared between two subsets (vectors) of criteria (maximization and minimization respectively), e.g., non-uniform VPMP are associated with two types of uniform problems.

According to these definitions, we will present a vector problem in mathematical programming with non-uniform criteria [6, 20, 22] in the following form:

$$Opt F(X) = \{\max F_1(X) = \{\max f_k(X), k = \overline{1, K_1}\}, \quad (1)$$

$$\min F_2(X) = \{\min f_k(X), k = \overline{1, K_2}\} \quad (2)$$

$$f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1, K}, G(X) \leq B, \quad (3)$$

$$X^{min} \leq X \leq X^{max} \quad (4)$$

where $X = \{x_j, j = \overline{1, N}\}$ is a vector of material variables, N -dimensional Euclidean space of R^N , $X^{min} \leq X \leq X^{max}$ - Parametric Constraints;

$F(X)$ is a vector function (vector criterion) having K – a component functions, (K - set power K), $F(X) = \{f_k(X), k = \overline{1, K}\}$. The set K consists of sets of K_1 , a component of maximization and K_2 of minimization; $K=K_1 \cup K_2$ therefore we enter the designation of the operation "opt," which includes *max* and *min*;

$F_1(X) = \{f_k(X), k = \overline{1, K_1}\}$ is maximizing vector-criterion, K_1 – number of criteria, and $K_1 = \overline{1, K_1}$ is a set of maximizing criteria (a problem (1), (3), (4) represents VPMP with the homogeneous maximizing criteria). Let's further assume that $f_k(X), k = \overline{1, K_1}$ is the continuous concave functions (we will sometimes call them the maximizing criteria);

$F_2(X) = \{f_k(X), k = \overline{1, K_2}\}$ is vector criterion in which each component is minimized, $K_2 = \overline{K_1 + 1, K} = \overline{1, K_2}$ - a set of minimization criteria, K_2 – number, (the problems (2)-(4) are VPMP with the homogeneous minimization criteria). We assume that $f_k(X), k = \overline{1, K_2}$ is the continuous convex functions:

$$K_1 \cup K_2 = K, K_1 \subset K, K_2 \subset K \quad (5)$$

f_k^{min}, f_k^{max} – Minimum and maximum value of the k th criterion in restrictions,

$G(X) \leq B, X \geq 0$ is standard restrictions, $g_i(X) \leq b_i, i = 1, \dots, M$ where b_i - a set of real numbers, and $g_i(X)$ are assumed continuous and convex.

$$S = \{X \in R^n | X \geq 0, G(X) \leq B, X^{min} \leq X \leq X^{max}\} \neq \emptyset, \quad (6)$$

where the set of admissible points set by restrictions (3)-(4) is not empty and represents a compact. The vector minimization function (criterion) $F_2(X)$ can be transformed to the vector maximization function (criterion) by the multiplication of each component of $F_2(X)$ to minus unit. The vector criterion of $F_2(X)$ is injected into VPMP (1)-(4) to show that, in a problem, there are two subsets of criteria of K_1, K_2 with, in essence, various directions of optimization.

We assume that the optimum points received by each criterion do not coincide for at least two criteria. If all points of an optimum coincide among themselves for all criteria, then we consider the decision trivially.



1.1.3. Axioms and Axiomatic Methods

An axiom is a statement that does not require logical proof. On the basis of these statements (initial assumptions), one or another theory is built.

The axiomatic method is a method of constructing a scientific theory, in which the theory is based on some initial assumptions called the axioms of the theory. As a result, all other provisions of the theory are obtained as logical consequences of axioms [1, 2].

In mathematics, the axiomatic method originated in the works of ancient Greek geometers. An example of the axiomatic method is the ancient Greek scientist Euclid, whose axioms were laid down in his famous work "Elements".

The axiomatic method was further developed in the works of D. Hilbert in the form of the so-called method of system formalism. The general scheme of building an arbitrary formal system ("S") includes:

1. The language of the system ("S"), including the alphabet – this is a list of elementary symbols; the rules of formation (syntax) on which the formulas "S" are built.
2. Axioms of the "S" system, which represent a certain set of formulas.
3. Rules for the withdrawal of the "S" system [2].

In the application to the solution of the problem of vector optimization (multidimensional mathematics), axiomatics is divided into two sections: 1. Axiomatics of solving the vector optimization problem with equivalent criteria; 2. Axiomatics of solving the vector optimization problem with a given priority of criteria. Only with the construction of the initial axiomatics is it possible in the future to construct the principle of optimality and the resulting algorithms for solving vector problems of mathematical programming.

1.2. Axiomatics, the principle of optimality and constructive methods of vector optimization with the equivalent criteria

1.2.1. Axiomatics of Solving a Vector Optimization Problem with Equivalent Criteria

In accordance with the above interpretation, the language of the system of multidimensional mathematics includes: first, the normalization of criteria, second, the relative evaluation of criteria (functions), and, third, the minimum relative level.

Definition 1. (Normalizing of the criterion).

Normalizing criteria (mathematical operation: the shift plus rationing) presents a unique display of the function $f_k(X) \forall k \in K$, in a one-dimensional space of $\mathbf{R}^1$ (the function $f_k(X) \forall k \in K$ represents a function of transformation from a N -dimensional Euclidean space of $\mathbf{R}^N$ in $\mathbf{R}^1$). To normalize criteria in vector problems, linear transformations will be used:

$$f_k(X) = a_k f'_k(X) + c_k \forall k \in K, \text{ or} \\ S = \{X \in \mathbf{R}^n | X \geq 0, G(X) \leq B, X^{\min} \leq X \leq X^{\max}\} \neq \emptyset, \quad (7)$$

where $f'_k(X)$, $k = \overline{1, K}$ - aged (before normalization) value of criterion; $f_k(X)$, $k = \overline{1, K}$ - the normalized value, a_k , c_k - constants.

Normalization of criteria $f_k(X) = (f'_k(X) + c_k)/a_k \forall k \in K$ is a simple (linear) invariant transformation of a polynomial, as a result of which the structure of the polynomial remains unchanged. In the optimization problem, the normalization of criteria $f_k(X) = (f'_k(X) + c_k)/a_k \forall k \in K$ does not affect the result of the solution. Indeed, if the convex optimization problem is solved:

$$\max_{X \in S} f(X), \text{ then at the optimum point } X^* \in S: \frac{df(X^*)}{dX} = 0. \quad (8)$$

In the general case (including the normalization of the criterion (1)), the problem is solved:

$$\max_{X \in S} (a_k f'_k(X) + c_k) \quad (9)$$

then at the optimum point $X^* \in S$:

$$\frac{d(a_k f(X^*) + c_k)}{dX} = a_k \frac{d(f(X^*))}{dX} + \frac{d(c_k)}{dX} = 0. \quad (10)$$



The result is identical, i.e. the optimum point X_k^* , $k = \overline{1, K}$ is the same for non-normalized and normalized problems.

Definition 2. Relative evaluation of the function (criterion).

In the vector problem (1)-(4), normalize (7) of the form:

$$\lambda_k(X) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}, \forall k \in K \quad (11)$$

is the relative estimate of a point $X \in S$ k th criterion $f_k(X)$ - k th criterion at the point $X \in S$; f_k^* - value of the k th criterion at the point of optimum X_k^* , obtained in vector problem (1) -(4) of individual k th criterion; f_k^0 is the worst value of the k th criterion (ant optimum) at the point X_k^0 (Superscript 0 - zero) on the admissible set S in vector problem (1)-(4);

the task at max (1), (3), (4) the value of f_k^0 is the lowest value of the k th criterion

$$f_k^0 = \min_{X \in S} f_k(X) \forall k \in K_1,$$

and task min (2), (3), (4) the value of f_k^0 is the greatest value of the k th criterion:

$$f_k^0 = \max_{X \in S} f_k(X) \forall k \in K_2.$$

The relative estimate of the $\lambda_k(X) \forall k \in K$ is first, measured in relative units; secondly, the relative assessment of the $\lambda_k(X) \forall k \in K$: on the admissible set is changed from zero in a point of X_k^0 : $\forall k \in K \lim_{X \rightarrow X_k^0} \lambda_k(X) = 0$, to the unit at the point of an optimum of X_k^* :

$$\forall k \in K \lim_{X \rightarrow X_k^*} \lambda_k(X) = 1: \forall k \in K 0 \leq \lambda_k(X) \leq 1, X \in S. \quad (12)$$

As a result of this normalization, all the criteria of the VPMP are (1)-(4) are comparable in relative units, which allows comparing them with each other, using criteria for joint optimization.

Definition 3. The operation of comparing relative estimates of a function (criterion) with each other.

Since any function (criterion) is represented in the relative estimates of the functions $\lambda_k(X) \forall k \in K$, which lie within the range of (12), it is possible to compare the relative estimates by numerical value. For comparison, the "subtraction" operation is used. If two functions (criteria) measured in the relative estimates $\lambda_{k=1}(X)$ and $\lambda_{k=2}(X) \forall k \in K$ are compared, then three situations are possible:

the first, when $\lambda_{k=1}(X) > \lambda_{k=2}(X)$;

the second, when $\lambda_{k=1}(X) = \lambda_{k=2}(X)$;

the third, when $\lambda_{k=1}(X) < \lambda_{k=2}(X)$.

The first and third situations are explored in Section 1.3.

This section examines the second situation.

1.2.2. Axiomatics of Vector Optimization with Equivalent Criteria

Axiom 1. (On the equivalence of criteria at an admissible point of a vector problem of mathematical programming). In of vector problems of mathematical programming two criteria with the indexes $k \in K, q \in K$ shall be considered equivalent in $X \in S$ point if relative estimates on k th and q th to criterion are equal among themselves in this point, i.e.

$$\lambda_k(X) = \lambda_q(X), k, q \in K.$$

Explanation. If at point $X \in S$ the functions (criteria) are equal to:

$\lambda_l(X) = 0,45 \ l \in K$ and $\lambda_q(X) = 0,45, q \in K$ (i.e., 45% of its optimal value, which in relative units is equal to 1), then such criteria are not "equal" to each other, but are equivalent in their numerical value. And each of them carries its own functional meaning, which can be obtained using the normalization of criteria (11).

Definition 4. (Definition of a minimum level among all relative estimates of criteria).

The relative level λ in a vector problem represents the lower assessment of a point of $X \in S$ among all relative estimates of $\lambda_k(X), k = \overline{1, K}$:



$$\forall X \in \mathbf{S} \lambda \leq \lambda_k(X), k = \overline{1, K}. \quad (13)$$

the lower level for performance of a condition (13) in an admissible point of $X \in \mathbf{S}$ is defined by a formula

$$\forall X \in \mathbf{S} \lambda = \min_{k \in K} \lambda_k(X). \quad (14)$$

Ratios (13) and (14) are interconnected. They serve as transition from operation (14) of definition of min to restrictions (13) and vice versa.

The level λ allows to unite all criteria in a vector problem one numerical characteristic of λ and to make over her certain operations, thereby, carrying out these operations over all criteria measured in relative units. The level λ functionally depends on the $X \in \mathbf{S}$ variable, changing X , we can change the lower level - λ . From here we will formulate the rule of search of the optimum decision.

1.2.3. The principle of optimality for vector optimization with the equivalent criteria

Definition 5. (The principle of an optimality 1 with equivalent criteria).

The vector problem of mathematical programming at equivalent criteria is solved, if the point of $X^o \in \mathbf{S}$ and a maximum level of λ^o (the top index o - optimum) among all relative estimates such that is found

$$\lambda^o = \max_{X \in \mathbf{S}} \min_{k \in K} \lambda_k(X). \quad (15)$$

Using interrelation of expressions (13) and (14), we will transform a maximine problem (15) to an extreme problem

$$\lambda^o = \max_{X \in \mathbf{S}} \lambda, \quad (16)$$

$$\text{at restriction } \lambda \leq \lambda_k(X), k = \overline{1, K} \quad (17)$$

The resulting problem (16)-(17) let's call the λ -problem.

λ -problem (16)-(17) has $(N+1)$ dimension, as a consequence of the result of the solution of λ -problem (16)-(17) represents an optimum vector of $\mathbf{X}^o \in R^{N+1}$, $(N+1)$ which component an essence of the value of the λ^o , i.e. $\mathbf{X}^o = \{x_1^o, x_2^o, \dots, x_N^o, x_{N+1}^o\}$, thus $x_{N+1}^o = \lambda^o$, and $(N+1)$ a component of a vector of $\mathbf{X}^o$ selected in view of its specificity.

The received a pair of $\{\lambda^o, X^o\} = \mathbf{X}^o$ characterizes the optimum solution of λ -problem (16)-(17) and according to vector problem of mathematical programming (1)-(4) with the equivalent criteria, solved on the basis of normalization of criteria and the principle of the guaranteed result. We will call in the optimum solution of $\mathbf{X}^o = \{\lambda^o, X^o\}$, X^o - an optimal point, and λ^o - a maximum level. An important result of the algorithm for solving vector problems (1)-(4) with equivalent criteria is the following theorem.

Theorem 1. (The theorem of the two contradictory criteria in the vector problem of mathematical programming with equivalent criteria).

In convex vector problems of mathematical programming (1)-(4) at the equivalent criteria which is solved on the basis of normalization of criteria and the principle of the guaranteed result, in an optimum point of $\mathbf{X}^o = \{\lambda^o, X^o\}$ two criteria are always - denote their indexes $q \in K, p \in K$ (which in a sense are the most contradiction of the criteria $k = \overline{1, K}$), for which equality is carried out:

$$\lambda^o = \lambda_q(X^o) = \lambda_p(X^o), q, p \in K, X \in \mathbf{S}, \quad (18)$$

and other criteria are defined by inequalities:

$$\lambda^o \leq \lambda_k(X^o), \forall k \in K, q \neq p \neq k. \quad (19)$$

For the first time, the proof of Theorem 1 is presented in [6, p. 22], and later it is repeated in [10, p.234]. Along with the fact that the point X^o is the optimal solution of the VPMP.

1.2.4. Mathematical algorithm of the solution of a vector problem with equivalent criteria

To solve of the vector problems of mathematical programming (1)-(4) the methods based on axiomatic of the normalization of criteria (11) and the principle of the guaranteed result (15), which follow from Axiom 1 and the principle of optimality 1. The constructive method for solving a vector optimization problem with equivalent criteria includes two blocks: the 1st block "System Analysis"



is divided into three steps; 2nd block "Optimal decision-making", which includes two steps: construction of the problem and its solution.

Block 1. System analysis.

Step 1. The problem (1)-(4) by each criterion separately is solved, i.e. for $\forall k \in K_1$ is solved at the maximum, and for $\forall k \in K_2$ is solved at a minimum. As a result of the decision, we will receive: X_k^* - an optimum point by the corresponding criterion, $k = \overline{1, K}$; $f_k^* = f_k(X_k^*)$ - the criterion size k th in this point, $k = \overline{1, K}$.

Step 2. We define the worst value of each criterion on S : f_k^0 , $k = \overline{1, K}$. For what the problem (1)-(4) for each criterion of $k = \overline{1, K_1}$ on a minimum is solved:

$$f_k^0 = \min f_k(X), G(X) \leq B, X \geq 0, k = \overline{1, K_1}.$$

The problem (1)-(4) for each criterion $k = \overline{1, K_2}$ maximum is solved:

$$f_k^0 = \max f_k(X), G(X) \leq B, X \geq 0, k = \overline{1, K_2}. \quad (20)$$

As a result of the decision, we will receive: $X_k^0 = \{x_j, j = \overline{1, N}\}$ - an optimum point by the corresponding criterion, $k = \overline{1, K}$;

$$f_k^0 = f_k(X_k^0) - \text{the criterion size } k\text{th a point, } X_k^0, k = \overline{1, K}.$$

Step 3. The system analysis of a set of points, optimum across Pareto, for this purpose in optimum points of $X^* = \{X_k^*, k = \overline{1, K}\}$, are defined sizes of criterion functions of $F(X^*)$ and relative estimates $\lambda(X^*)$, $\lambda_k(X) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}$, $\forall k \in K$:

$$F(X^*) = \{f_k(X_k^*), q = \overline{1, K}, k = \overline{1, K}\} = \begin{bmatrix} f_1(X_1^*), \dots, f_K(X_1^*) \\ \dots \\ f_1(X_K^*), \dots, f_K(X_K^*) \end{bmatrix}, \quad (21)$$

$$\lambda(X^*) = \{\lambda_q(X_k^*), q = \overline{1, K}, k = \overline{1, K}\} = \begin{bmatrix} \lambda_1(X_1^*), \dots, \lambda_K(X_1^*) \\ \dots \\ \lambda_1(X_K^*), \dots, \lambda_K(X_K^*) \end{bmatrix}. \quad (22)$$

As a whole on a problem $\forall k \in K$ the relative assessment of $\lambda_k(X)$, $k = \overline{1, K}$ (22) lies within $0 \leq \lambda_k(X) \leq 1, k = \overline{1, K}$.

Block 2. Making the optimal decision in the VPMP. It includes two steps - 4, 5.

Step 4. Creation of the λ -problem.

Creation of λ -problem is carried out in two stages:

initially built the maximine problem of optimization with the normalized criteria which at the second stage will be transformed to the standard problem of mathematical programming called λ -problem.

For construction maximine a problem of optimization we use definition 2 - relative level:

$$\forall X \in S \quad \lambda = \min_{k \in K} \lambda_k(X).$$

The bottom λ level is maximized on $X \in S$, as a result we will receive a maximine problem of optimization with the normalized criteria.

$$\lambda^0 = \max_{X \in S} \min_{k \in K} \lambda_k(X). \quad (23)$$

At the second stage we will transform a problem (23) to a standard problem of mathematical programming:

$$\lambda^0 = \max_{X \in S} \lambda, \quad \rightarrow \quad \lambda^0 = \max_{X \in S} \lambda. \quad (24)$$

$$\lambda - \lambda_k(X) \leq 0, k = \overline{1, K}, \quad \rightarrow \quad \lambda - \frac{f_k(X) - f_k^0}{f_k^* - f_k^0} \leq 0, k = \overline{1, K}, \quad (25)$$

$$G(X) \leq B, X \geq 0, \quad \rightarrow \quad G(X) \leq B, X \geq 0, \quad (26)$$

where the vector of unknown of X has dimension of $N + 1$: $X = \{\lambda, x_1, \dots, x_N\}$.

Step 5. Solution of λ -problem.

λ -problem (24)-(26) is a standard problem of convex programming and for its decision standard methods are used. As a result of the solution of λ -problem it is received:



$$X^o = \{X^o, \lambda^o\} - \text{an optimum point}; \quad (27)$$

$$f_k(X^o), k = \overline{1, K} \text{ are values of the criteria in this point}; \quad (28)$$

$$\lambda_k(X^o) = \frac{f_k(X) - f_k^o}{f_k^* - f_k^o}, k = \overline{1, K} \text{ are sizes of relative estimates}. \quad (29)$$

λ^o – the maximum relative estimates which is the maximum bottom level for all relative estimates of $\lambda_k(X^o)$, or the guaranteed result in relative units. λ^o guarantees that all relative estimates of $\lambda_k(X^o)$ more or are equal λ^o :

$$\lambda_k(X^o) \geq \lambda^o, k = \overline{1, K} \text{ or } \lambda^o \leq \lambda_k(X^o), k = \overline{1, K}, X^o \in S, \quad (30)$$

and according to the theorem 1 point of $X^o = \{\lambda^o, x_1, \dots, x_N\}$ is optimum across Pareto.

1.3. The axioms and the principle of optimality for vector optimization with a Criterion Priority

Definition 3 states that if we compare two functions (criteria) measured in relative estimates $\lambda_{k=1}(X)$ and $\lambda_{k=2}(X) \forall k \in K$, then three situations are possible. The second situation, when $\lambda_{k=1}(X) = \lambda_{k=2}(X)$ is investigated in Section 1.2 (equivalent criteria). Situations: the first, when $\lambda_{k=1}(X) > \lambda_{k=2}(X)$, and the third, when $\lambda_{k=1}(X) < \lambda_{k=2}(X)$, are explored in this section. Such situations are defined as tasks with the priority of the criterion.

For development of methods of the solution of problems of vector optimization with a priority of criterion we use definitions as follows: Priority of one criterion of vector problems, with a criterion priority over other criteria; Numerical expression of a priority; The set priority of a criterion; the lower (minimum) level from all criteria with a priority of one of them; a subset of points with priority by criterion (Axiom 2); the principle of optimality of the solution of problems of vector optimization with the set priority of one of the criteria, and related theorems. For more details see [20, 22].

1.3.1. Axiomatics of solving a Vector Optimization Problem with a given criterion priority

The language of the axiomatics system for solving a vector problem with a given criterion priority includes definitions: 1) Priority of one criterion over another; 2) The numerical value of the priority of the criterion; 3) The lowest level of the criterion among all relative evaluations with the priority of the criterion.

Definition 6. (About the priority of one criterion over the other).

The criterion of $q \in K$ in the vector problem of Equations (1)-(4) in a point of $X \in S$ has priority over other criteria of $k = \overline{1, K}$, and the relative estimate of $\lambda_q(X)$ by this criterion is greater than or equal to relative estimates of $\lambda_k(X)$ of other criteria, i.e.:

$$\lambda_q(X) \geq \lambda_k(X), k = \overline{1, K}, \quad (31)$$

and a strict priority for at least one criterion of $t \in K$,

$$\lambda_q(X) > \lambda_k(X), t \neq q, \text{ and for other criteria of } \lambda_q(X) \geq \lambda_k(X), k = \overline{1, K}, k \neq t \neq q.$$

Introduction of the definition of a priority of criterion $q \in K$ in the vector problem of Equations (1)-(4) executed the redefinition of the early concept of a priority. Earlier the intuitive concept of the importance of this criterion was outlined, now this "importance" is defined as a mathematical concept: the higher the relative estimate of the q th criterion compared to others, the more it is important (i.e., more priority), and the highest priority at a point of an optimum is $X_k^*, \forall q \in K$.

From the definition of a priority of criterion of $q \in K$ in the vector problem of Equations (1)-(4), it follows that it is possible to reveal a set of points $S_q \subset S$ that is characterized by

$\lambda_q(X) \geq \lambda_k(X), \forall k \neq q, \forall X \in S_q$. However, the answer to whether a criterion of $q \in K$ at a point of the set S_q has more priority than others do remains open. For clarification of this question, we define a communication coefficient between a couple of relative estimates of q and k that, in total, represent a vector:

$$P^q(X) = \{p_k^q(X) | k = \overline{1, K}\}, q \in K \forall X \in S_q.$$



Definition 7. (About numerical expression of a priority of one criterion over another).

In the vector problem of Equations (1) and (4), with priority of the q th criterion over other criteria of $k = \overline{1, K}$, for $\forall X \in S_q$, and a vector of $P^q(X)$ which shows how many times a relative estimate of $\lambda_q(X)$, $q \in K$, is more than other relative estimates of $\lambda_k(X)$, $k = \overline{1, K}$, we define a numerical expression of the priority of the q th criterion over other criteria of $k = \overline{1, K}$ as:

$$P^q(X) = \{p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}, k = \overline{1, K}\}, p_k^q(X) \geq 1, \forall X \in S_q \subset S, k = \overline{1, K}, \forall q \in K. \quad (32)$$

Such a ratio $p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}$, let us call *the numerical expression of the priority of the q -th criterion over the rest of the criteria $k = \overline{1, K}$* .

Definition 7a. (On a given numerical expression of the priority of one criterion over others).

In the vector problem of Equations (1)–(4) with a priority of criterion of $q \in K$ for $\forall X \in S$, vector $P^q = \{p_k^q, k = \overline{1, K}\}$ is considered to be set by the person making decisions (i.e., decision-maker) if everyone is set a component of this vector. Set by the decision-maker, component p_k^q , from the point of view of the decision-maker, shows how many times a relative estimate of $\lambda_q(X)$, $q \in K$ is greater than other relative estimates of $\lambda_k(X)$, $k = \overline{1, K}$. The vector of $p_k^q, k = \overline{1, K}$, is the numerical expression of the priority of the q th criterion over other criteria of $k = \overline{1, K}$:

$$P^q(X) = \{p_k^q(X), k = \overline{1, K}\}, p_k^q(X) \geq 1, \forall X \in S_q \subset S, k = \overline{1, K}, \forall q \in K. \quad (33)$$

The vector problem of Equations (1)–(4), in which the priority of any criteria is set, is called a vector problem with the set priority of criterion. The problem of a task of a vector of priorities arises when it is necessary to determine the point $X^o \in S$ by the set vector of priorities. In the comparison of relative estimates with a priority of criterion of $q \in K$, as well as in a task with equivalent criteria, we define the additional numerical characteristic of λ which we call the level.

Definition 8. (About the lower level among all relative estimates with a criterion priority).

The λ level is the lowest among all relative estimates with a priority of criterion of $q \in K$ such that:

$$\lambda \leq p_k^q \lambda_k(X), k = \overline{1, K}, q \in K, \forall X \in S_q \subset S. \quad (34)$$

The lower level for the performance of the condition in Equation (34) is defined as:

$$\lambda = \min_{k \in K} p_k^q \lambda_k(X), q \in K, \forall X \in S_q \subset S. \quad (35)$$

Equations (34) and (35) are interconnected and serve as a further transition from the operation of the definition of the minimum to restrictions, and vice versa. In Section 1.2.4, we gave the definition of a Pareto optimal point $X^o \in S$ with equivalent criteria. Considering this definition as an initial one, we will construct a number of the axioms dividing an admissible set of S into, first, a subset of Pareto optimal points S^o , and, secondly, a subset of points $S_q \subset S, q \in K$, with priority for the q th criterion.

1.3.2. The axioms for vector optimization with a Criterion Priority

Axiom 2. (About a subset of points, priority by criterion).

In the vector problem of Equations (1)–(4), the subset of points $S_q \subset S$ is called the area of priority of criterion of $q \in K$ over other criteria, if

$$\forall X \in S_q \quad \forall k \in K \quad \lambda_q(X) \geq \lambda_k(X), q \neq k.$$

This definition extends to a set of Pareto optimal points S^o that is given by the following definition.

Axiom 2a. (About a subset of points, priority by criterion, on Pareto's great number in a vector problem).

In a vector problem of mathematical programming (1)–(4) the subset of points $S_q^o \subset S^o \subset S$ is called the area of a priority of criterion of $q \in K$ over other criteria, if



$$\forall X \in S_q^o \forall k \in K \lambda_q(X) \geq \lambda_k(X), q \neq k.$$

In the following we provide explanations.

Axiom 2 and 2a allow the breaking of the vector problem in Equations (1)–(4) into an admissible set of points S , including a subset of Pareto optimal points, $S^o \subset S$, and subsets:

One subset of points $S' \subset S$ where criteria are equivalent, and a subset of points of S' crossed with a subset of points S^o , allocated to a subset of Pareto optimal points at equivalent criteria $S^{oo} = S' \cap S^o$. As will be shown further, this consists of one point of $X^o \in S$, i.e.

$$X^o = S^{oo} = S' \cap S^o, S' \in S, S^o \subset S.$$

"K" subsets of points where each criterion of $q = \overline{1, K}$ has a priority over other criteria of $k = \overline{1, K}, q \neq k$, and thus breaks, first, sets of all admissible points S , into subsets $S_q \subset S, q = \overline{1, K}$ and, second, a set of Pareto optimal points, S^o , into subsets $S_q^o \subset S_q \subset S, q = \overline{1, K}$. This yields:

$$(S' \cup (\bigcup_{q \in K} S_q^o)) \equiv S^o, S_q^o \subset S^o \subset S, q = \overline{1, K}.$$

We note that the subset of points S_q^o , on the one hand, is included in the area (a subset of points) of priority of criterion of $q \in K$ over other criteria: $S_q^o \subset S_q \subset S$, and, on the other, in a subset of Pareto optimal points $S_q^o \subset S^o \subset S$.

Axiom 2 and the numerical expression of priority of criterion (Definition 5) allow the identification of each admissible point of $X \in S$ (by means of vector:

$$P^q(X) = \{p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}, k = \overline{1, K}\}, \text{ to form and choose:}$$

a subset of points by priority criterion S_q , which is included in a set of points S ,

$\forall q \in K X \in S_q \subset S$, (such a subset of points can be used in problems of clustering, but is beyond this article);

a subset of points by priority criterion S_q^o , which is included in a set of Pareto optimal points $S^o, \forall q \in K, X \in S_q^o \subset S^o$.

Thus, full identification of all points in the vector problem of Equations (1)–(4) is executed in sequence as:

Set of admissible points of $X \in S \rightarrow$	Subset of points, optimum across Pareto $X \in S^o \subset S \rightarrow$	Subset of points, optimum across Pareto $X \in S_q^o \subset S^o \subset S \rightarrow$	Separate point of a $\forall X \in S$ $X \in S_q^o \subset S^o \subset S$
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This is the most important result which allows the output of the principle of optimality and to construct methods of a choice of any point of Pareto's great number.

1.3.3. The principle of optimality for vector optimization with a Criterion Priority

Definition 9. (Principle of optimality 2. The solution of a vector problem with the set criterion priority).

The vector problem of Equations (1)–(4) with the set priority of the q th criterion of $p_k^q \lambda_k(X), k = \overline{1, K}$ is considered solved if the point X^o and maximum level λ^o among all relative estimates is found such that:

$$\lambda^o = \max_{X \in S} \min_{k \in K} p_k^q \lambda_k(X), q \in K. \quad (36)$$

Using the interrelation of Equations (34) and (35), we can transform the maximine problem of Equation (36) into an extreme problem of the form:

$$\lambda^o = \max_{X \in S} \lambda. \quad (37)$$

$$\text{at restriction } \lambda \leq p_k^q \lambda_k(X), k = \overline{1, K}. \quad (38)$$

We call Equations (37) and (38) the λ -problem with a priority of the q th criterion.



The solution of the λ -problem is the point $\mathbf{X}^o = \{X^o, \lambda^o\}$. This is also the result of the solution of the vector problem of Equations (1)-(4) with the set priority of the criterion, solved on the basis of normalization of criteria and the principle of the guaranteed result.

In the optimum solution $\mathbf{X}^o = \{X^o, \lambda^o\}$, X^o , an optimum point, and λ^o , the maximum bottom level, the point of X^o and the λ^o level correspond to restrictions of Equation (4), which can be written as: $\lambda^o \leq p_k^q \lambda_k(X^o), k = \overline{1, K}$.

These restrictions are the basis of an assessment of the correctness of the results of a decision in practical vector problems of optimization.

From Definitions 1 and 2, "Principles of optimality", follows the opportunity to formulate the concept of the operation "opt".

Definition 9. (Mathematical operation "opt").

In the vector problem of Equations (1)–(4), in which "max" and "min" are part of the criteria, the mathematical operation "opt" consists of the definition of a point X^o and the maximum λ^o bottom level to which all criteria measured in relative units are lifted:

$$\lambda^o \leq \lambda_k(X^o) = \frac{f_k(X^o) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}, \quad (39)$$

i.e., all criteria of $\lambda_k(X^o), k = \overline{1, K}$, are equal to or greater than the maximum level of λ^o (therefore λ^o is also called the guaranteed result).

Theorem 2. (The theorem of the most inconsistent criteria in a vector problem with the set priority).

If in the convex vector problem of mathematical programming of Equations (1)–(4) the priority of the q th criterion of $p_k^q, k = \overline{1, K}, \forall q \in \mathbf{K}$ over other criteria is set, at a point of an optimum $X^o \in \mathbf{S}$ obtained on the basis of normalization of criteria and the principle of guaranteed result, there will always be two criteria with the indexes $r \in \mathbf{K}, t \in \mathbf{K}$, for which the following strict equality holds:

$$\lambda^o = p_k^r \lambda_r(X^o) = p_k^t \lambda_t(X^o), r, t \in \mathbf{K}, \quad (40)$$

and other criteria are defined by inequalities:

$$\lambda^o \leq p_k^q \lambda_k(X^o), k = \overline{1, K}, \forall q \in \mathbf{K}, q \neq r \neq t. \quad (41)$$

Criteria with the indexes $r \in \mathbf{K}, t \in \mathbf{K}$ for which the equality of Equation (41) holds are called the most inconsistent.

Proof. Similar to Theorem 2 [20].

We note that in Equations (40) and (41), the indexes of criteria $r, t \in \mathbf{K}$ can coincide with the $q \in \mathbf{K}$ index.

Consequence of Theorem 1, about equality of an optimum level and relative estimates in a vector problem with two criteria with a priority of one of them.

In a convex vector problem of mathematical programming with two equivalent criteria, solved on the basis of normalization of criteria and the principle of the guaranteed result, at an optimum point X^o equality is always carried out at a priority of the first criterion over the second:

$$\lambda^o = \lambda_1(X^o) = p_2^1(X^o) \lambda_2(X^o), X^o \in \mathbf{S}, \quad (42)$$

where $p_2^1(X^o) = \lambda_1(X^o) / \lambda_2(X^o)$, and at a priority of the second criterion over the first: $\lambda^o = \lambda_2(X^o) = p_1^2(X^o) \lambda_1(X^o), X^o \in \mathbf{S}$, where $p_1^2(X^o) = \lambda_2(X^o) / \lambda_1(X^o)$.

1.3.4. Mathematical Method of the Solution of a Vector Problem with Criterion Priority

Step 1. We solve a vector problem with equivalent criteria. The algorithm of the decision is presented in Section 4.4.

As a result of the decision, we obtain:

optimum points by each criterion separately $X_k^*, k = \overline{1, K}$ and sizes of criterion functions in these points of $f_k^* = f_k(X_k^*), k = \overline{1, K}$, which represent the boundary of a set of Pareto optimal points; anti-optimum points by each criterion of $X_k^0 = \{x_j, j = \overline{1, N}\}$ and the worst unchangeable part of each criterion of $f_k^0 = f_k(X_k^0), k = \overline{1, K}$;



$X^o = \{X^o, \lambda^o\}$, an optimum point, as a result of the solution of VPMP at equivalent criteria, i.e., the result of the solution of a maximine problem and the λ -problem constructed on its basis;

λ^o , the maximum relative assessment which is the maximum lower level for all relative estimates of $\lambda_k(X^o)$, or the guaranteed result in relative units, λ^o guarantees that all relative estimates of $\lambda_k(X^o)$ are equal to or greater than λ^o :

$$\lambda^o \leq \lambda_k(X^o), k = \overline{1, K}, X^o \in S. \quad (43)$$

The person making the decision carries out the analysis of the results of the solution of the vector problem with equivalent criteria.

If the received results satisfy the decision maker, then the process concludes, otherwise subsequent calculations are performed.

In addition, we calculate:

in each point $X_k^*, k = \overline{1, K}$ we determine sizes of all criteria of: $q = \overline{1, K}$

$\{f_q(X_k^*), q = \overline{1, K}\}, k = \overline{1, K}$, and relative estimates

$\lambda(X^*) = \{\lambda_q(X_k^*), q = \overline{1, K}, k = \overline{1, K}\}, \lambda_k(X) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}, \forall k \in K$:

$$(X^*) = \begin{bmatrix} f_1(X_1^*) & \dots & f_K(X_1^*) \\ \dots & \dots & \dots \\ f_1(X_K^*) & \dots & f_K(X_K^*) \end{bmatrix}, \lambda(X^*) = \begin{bmatrix} \lambda_1(X_1^*) & \dots & \lambda_K(X_1^*) \\ \dots & \dots & \dots \\ \lambda_1(X_K^*) & \dots & \lambda_K(X_K^*) \end{bmatrix}. \quad (44)$$

Matrices of criteria of $F(X^*)$ and relative estimates of $\lambda(X^*)$ show the sizes of each criterion of $k = \overline{1, K}$ upon transition from one optimum point $X_k^*, k \in K$ to another $X_q^*, q \in K$, i.e., on the border of a great number of Pareto.

At an optimum point at equivalent criteria X^o we calculate sizes of criteria and relative estimates:

$$f_k(X^o), k = \overline{1, K}; \lambda_k(X^o), k = \overline{1, K}, \quad (45)$$

which satisfy the inequality of Equation (43). In other points $X \in S^o$, in relative units the criteria of $\lambda = \min_{k \in K} \lambda_k(X)$ are always less than λ^o , given the λ -problem of Equations (24)-(26). This information is also a basis for further study of the structure of a great number of Pareto.

Step 2. Choice of priority criterion of $q \in K$.

From theory (see Theorem 1) it is known that at an optimum point X^o there are always two most inconsistent criteria, $q \in K$ and $v \in K$, for which in relative units an exact equality holds:

$\lambda^o = \lambda_q(X^o) = \lambda_v(X^o), q, v \in K, X \in S$. Others are subject to inequalities:

$\lambda^o \leq \lambda_k(X^o), \forall k \in K, q \neq v \neq k$.

As a rule, the criterion which the decision-maker would like to improve is part of this couple, and such a criterion is called a priority criterion, which we designate $q \in K$.

Step 3. Numerical limits of the change of the size of a priority of criterion $q \in K$ are defined.

For priority criterion $q \in K$ from the matrix of Equation (40) we define the numerical limits of the change of the size of criterion:

$$\text{in physical units of } f_k(X^o) \leq f_q(X) \leq f_q(X_q^*), k \in K, \quad (46)$$

where $f_q(X_q^*)$ derives from the matrix of Equation $F(X^*)$ (44), all criteria showing sizes measured in physical units, $f_k(X^o), k = \overline{1, K}$ from Equation (45), and,

$$\text{in relative units of } \lambda_k(X^o) \leq \lambda_q(X) \leq \lambda_q(X_q^*), k \in K, \quad (47)$$

where $\lambda_q(X_q^*)$ derives from the matrix $\lambda(X^*)$, all criteria showing sizes measured in relative units (we note that $\lambda_q(X_q^*) = 1$), $\lambda_q(X^o)$ from Equation (44).

As a rule, Equations (46) and (47) are given for the display of the analysis.

Step 4. Choice of the size of priority criterion (decision-making).

The person making the decision carries out the analysis of the results of calculations of Equation (44) and from the inequality of Equation (46) chooses the numerical size f_q of the criterion of $q \in K$:



$$\text{in relative units of } \lambda_k(X^o) \leq \lambda_q(X) \leq \lambda_q(X_q^*), k \in K, \quad (48)$$

For the chosen size of the criterion of f_q it is necessary to define a vector of unknown X^o . For this purpose, we carry out the subsequent calculations.

Step 5. Calculation of a relative assessment.

For the chosen size of the priority criterion of f_q the relative assessment is calculated as:

$\lambda_q = \frac{f_q - f_q^0}{f_q^* - f_q^0}$, which upon transition from point X^o to X_q^* , according to Equation (43), lies in the limits: $\lambda_q(X^o) \leq \lambda_q \leq \lambda_q(X_q^*) = 1$.

Step 6. Calculation of the coefficient of linear approximation.

Assuming a linear nature of the change of criterion of $f_q(X)$ in Equation (48) and according to the relative assessment of $\lambda_q(X)$, using standard methods of linear approximation we calculate the proportionality coefficient between $\lambda_q(X^o)$, λ_q , which we call ρ : $\rho = \frac{\lambda_q - \lambda_q(X^o)}{\lambda_q^* - \lambda_q^0}$, $q \in K$.

Step 7. Calculation of coordinates of priority criterion with the size f_q .

In accordance with Equation (44), the coordinates of the X_q priority criterion point lie within the following limits: $X^o \leq X_q \leq X_q^*$, $q \in K$. Assuming a linear nature of change of the vector

$X_q = \{x_1^q, \dots, x_N^q\}$ we determine coordinates of a point of priority criterion with the size f_q with the relative assessment of Equation (45):

$$\begin{aligned} X_q &= \{x_1^q = x_1^o + \rho(x_1^*(1) - x_1^o), \dots, \\ &\quad x_N^q = x_N^o + \rho(x_N^*(N) - x_N^o)\}, \end{aligned} \quad (49)$$

where $X^o = \{x_1^o, \dots, x_N^o\}$, $X_q^* = \{x_q^*(1), \dots, x_q^*(N)\}$.

Step 8. Calculation of the main indicators of a point x_q (49).

For the obtained point x_q , we calculate:

all criteria in physical units: $F^q = \{f_k(x^q), k = \overline{1, K}\}$;

all relative estimates of criteria:

$$\lambda^q = \{\lambda_k^q, k = \overline{1, K}\}, \lambda_k(x^q) = \frac{f_k(x^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}; \quad (50)$$

the vector of priorities: $P^q = \{p_k^q = \frac{\lambda_k(x^q)}{\lambda_k(x^q)}, k = \overline{1, K}\}$;

the maximum relative assessment: $\lambda^{oq} = \min(p_k^q \lambda_k(x^q), k = \overline{1, K})$.

Any (50) point from Pareto's set $X_t^o = \{\lambda_t^o, X_t^o\} \in S^o$ can be similarly calculated.

Analysis of results. The calculated size of criterion $f_q(X_t^o)$, $q \in K$ is usually not equal to the set f_q . The error of the choice of $\Delta f_q = |f_q(X_t^o) - f_q|$ is defined by the error of linear approximation. The results of the study of *symmetry* in VPMP with a given priority are similar as for VPMP with equivalent criteria, but the center of symmetry is shifted towards the priority criterion.

1.4. Theoretical Analysis of the Maximin Problem, Comparison λ -Criterion with Weighting Coefficients in the Solution of a Vector Problem

Theoretical analysis of the maximin problem. In structure, problem (15, 36) appears to be no different from the analogous problem considered in a series of papers¹:

$$\lambda^o = \max_{X \in S} \min_{k \in K} \{w_k, \lambda_k(X)\}, X \in S, X \geq 0, \quad (51)$$

where $\lambda_k(X) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}$ is the relative estimate is the relative estimate; f_k^* and f_k^0 are the optimal and worst values of the k th criterion, respectively (i.e., the investigation is performed in relative units

¹ V.S. Mikhailevich and V.L. Volkovich, Numerical Methods for the Investigation and Design of Complex Systems (Nauka, Moscow, 1979) [in Russian]. V.S. Mikhailevich, V.L. Volkovich, and G.V. Kolenov, "Algorithm for coordinating solutions in a distributed system of interconnected problems with linear models," Kibernetika, no. 3, 1-8, 22 (1988).



$\lambda_k(X)$; w_k are the weighting coefficients which satisfy the ordinary conditions $w_k > 0$, $w_1 + \dots + w_k = 1$ and express the preference of the criteria. The preference is determined by the fact that for a preferred (priority) criterion a greater weight is assigned.

However, internally, problems (36) and (51) fundamentally differ from one another. We demonstrate this difference with the two-criterion maximization problem:

$$f_1(X) \rightarrow \max, f_2(X) \rightarrow \max, \quad (52)$$

$$g(X) \leq b, X \geq 0, X = \{x_1, x_2\}, x_1 \geq 0, x_2 \geq 0, \quad (53)$$

where the set of admissible points S described by constraints (53) is not empty $S \neq \emptyset$ and represents a compact set. Using vector problem (52) and (53) as the base, we construct the maximin problem with normalized criteria:

$$\lambda^0 = \max_{X \in S} \min_{k \in K} (\lambda_1(X), \lambda_2(X)), X \in S, X \geq 0,$$

where $\lambda_1(X)$ and $\lambda_2(X)$ are relative estimates for the first and second criteria, while X_1^* and X_2^* are the optimal points obtained in the solution of VPMP (52) and (53) for the first and second criteria, respectively. We transform the maximin problem into the λ -problem:

$$\lambda^0 = \max_{X \in S} \lambda, \lambda - \lambda_1(X) \leq 0, \lambda - \lambda_2(X) \leq 0, X \in S, X \geq 0.$$

We present these inequalities as equalities by adding the variables x' and x'' . Then we subtract one of the expressions in question from another: $\lambda_1(X) - x' = \lambda_2(X) - x''$. The additional variables x' and x'' for the two-criterion problem are balanced and at the optimal point are equal to zero (this equality is confirmed by the above theorem at stage 2); as a result, we have

$$\lambda_1(X^0) = \lambda_2(X^0), \quad (54)$$

Therefore, the algorithm is said to be "with equivalent criteria."

Consider the change in equality (54) with the introduction of the vector of priorities and the weight vector. We assume that the first criterion has a twofold priority over the second criterion. Then, the vector of priorities is presented as $P^1 = \{p_1^1 = 1, p_2^1 = 2\}$, and the weight vector has the form: $w = \{w_1 = 2/3, w_2 = 1/3\}, w_1 + w_2 = 1$.

Here, equality (54) with the vector of priorities P^1 takes the form:

$$p_1^1 \lambda_1(X^0) = p_2^1 \lambda_2(X^0) \text{ or } \lambda_1(X^0) = 2\lambda_2(X^0) \quad (55)$$

and with the weight vector, in contrast:

$$w_1 \lambda_1(X^0) = w_2 \lambda_2(X^0) \text{ или } 2\lambda_1(X^0)/3 = \lambda_2(X^0)/3 \text{ или } 2\lambda_1(X^0) = \lambda_2(X^0). \quad (56)$$

As a result of solving VPMP (52) and (53) with a vector of priorities (55), the relative estimate for the first criterion $\lambda_1(X^0)$ will be twice as large as the relative estimate for the second criterion $\lambda_2(X^0)$, and this is required according to the formulation of the problem.

In solving VPMP (52) and (53) with the weight vector (56), the relative estimate for the first criterion $\lambda_1(X^0)$ will be half the relative estimate for the second criterion $\lambda_2(X^0)$, and this is in complete contradiction with the formulation of the problem.

We present this analysis by the example of a VPMP with two criteria.

Example. Consider a homogeneous VPMP (in particular, a maximization problem)

$$f_1(X) = 5x_1 + x_2 \rightarrow \max, \quad f_2(X) = 20x_1 + 80x_2 \rightarrow \max, \quad (57)$$

$$5x_1 + 5x_2 \leq 125, x_1 \leq 28, x_2 \leq 20, x_1 \geq 0, x_2 \geq 0. \quad (58)$$

The form of the admissible set of solutions controlled by constraints (58), is shown in Fig. 1.



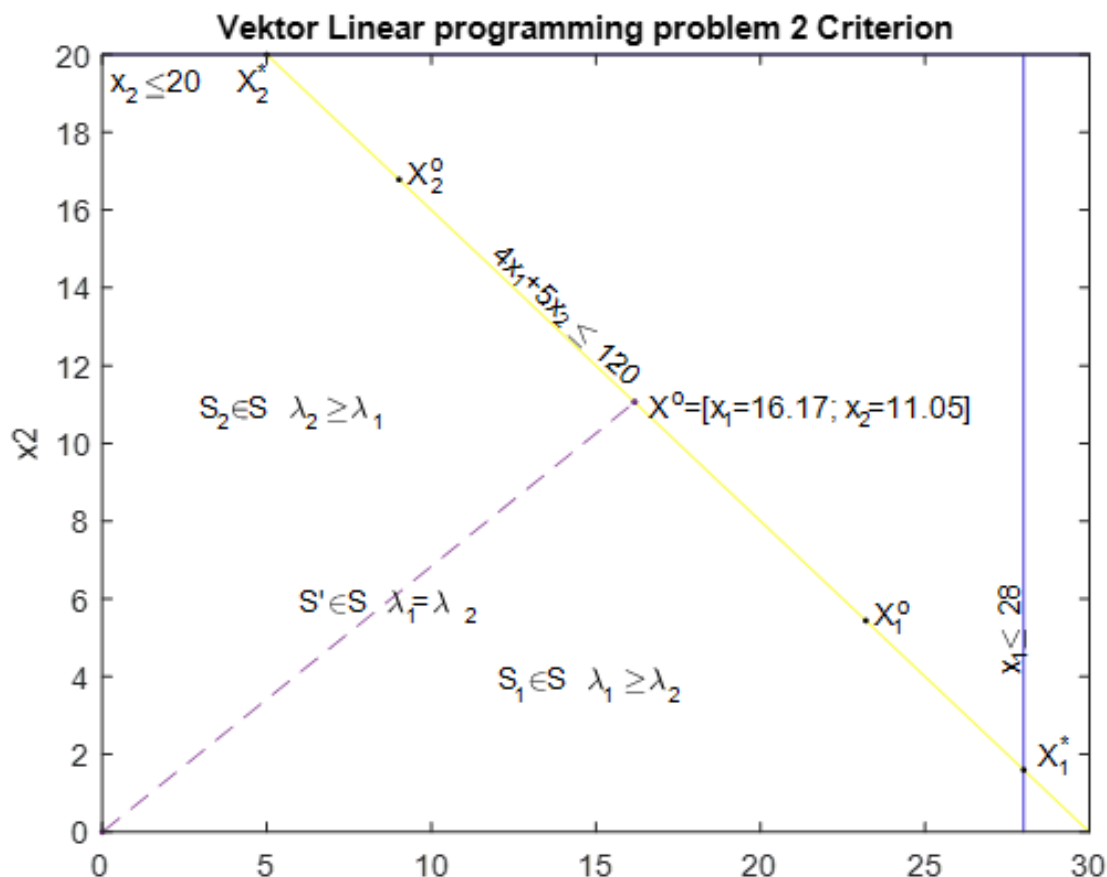


Figure 1. Solution of VPMP (57) and (58)

We present the solution of VPMP (57) and (58) in the MATLAB system, according to the algorithm for solving a VPMP based on the normalization of the criteria and the principle of a guaranteed result. In order to obtain the solution for each criterion, we use the `linprog` function, which solves the linear programming problem.

Step 1. Decision of the VPMP (57) -(58) for each criterion separately (57) -(58). As a result of the calculation, we get the optimum points, X_k^* and f_k^* , $k = \overline{1,2}$, Fig. 1.

$$X_1^* = \{x_1 = 28, x_2 = 1.6\}, \quad f_1^* = f_1(X_1^*) = 141.6;$$

Step 2. is not executed. For a VPMP with weak constraints, the worst values of the criteria are $f_1^0 = f_2^0 = 0$. As a result, the normalization takes the form:

$$\lambda_k(X) = f_k(X)/f_k^*, \quad k = \overline{1,2}.$$

Step 3. The criteria are analyzed; for this purpose, we determine the matrices of the targeted functions and the relative estimates at the optimal points X_1^*, X_2^*

$$F(X^*) = \begin{bmatrix} 141.6 & 752.0 \\ 45.0 & 2500 \end{bmatrix}, \quad \lambda(X^*) = \begin{bmatrix} 1.0 & 0.3008 \\ 0.0450 & 1.0 \end{bmatrix}.$$

Step 4. We construct the λ -problem:

$$\begin{aligned} \lambda^0 &= \max \lambda, \\ \lambda - (5x_1 + 1x_2)/141.6 &\leq 0, \lambda - (20x_1 + 80x_2)/2500 \leq 0, \\ x_1 + 5x_2 &\leq 125, x_1 \leq 28, x_2 \leq 20, x_1 \geq 0, x_2 \geq 0, \end{aligned} \quad (59)$$

which is solved using the `linprog` function. The results represent the optimal values of the variables $X^0 = \{\lambda = 0.6547, x_1 = 16.3585, x_2 = 10.9132\}$; and the optimal value of the goal function:



$\lambda^o = 0.6547$. The obtained functions $\{\lambda_1(X^o)\}$ and $\{\lambda_2(X^o)\}$, together with the optimal points X^o and λ^o of their intersection, are presented in Fig. 2.

We perform a check: $f_1(X^o) = 92.7$, $\lambda_1(X^o) = 0.6547$; $f_2(X^o) = 1636.8$ and $\lambda_2(X^o) = 0.6547$, i.e., $\lambda^o \leq \lambda_k(X^o)$, $k = 1, 2$. These results show that at the point X^o , both criteria in relative units attain the level $\lambda^o = 0.6547$ of their optimal values. Any increase in one of the criteria above this level causes a decrease in the other criterion; i.e., X^o is the optimal Pareto point (see Fig. 2).

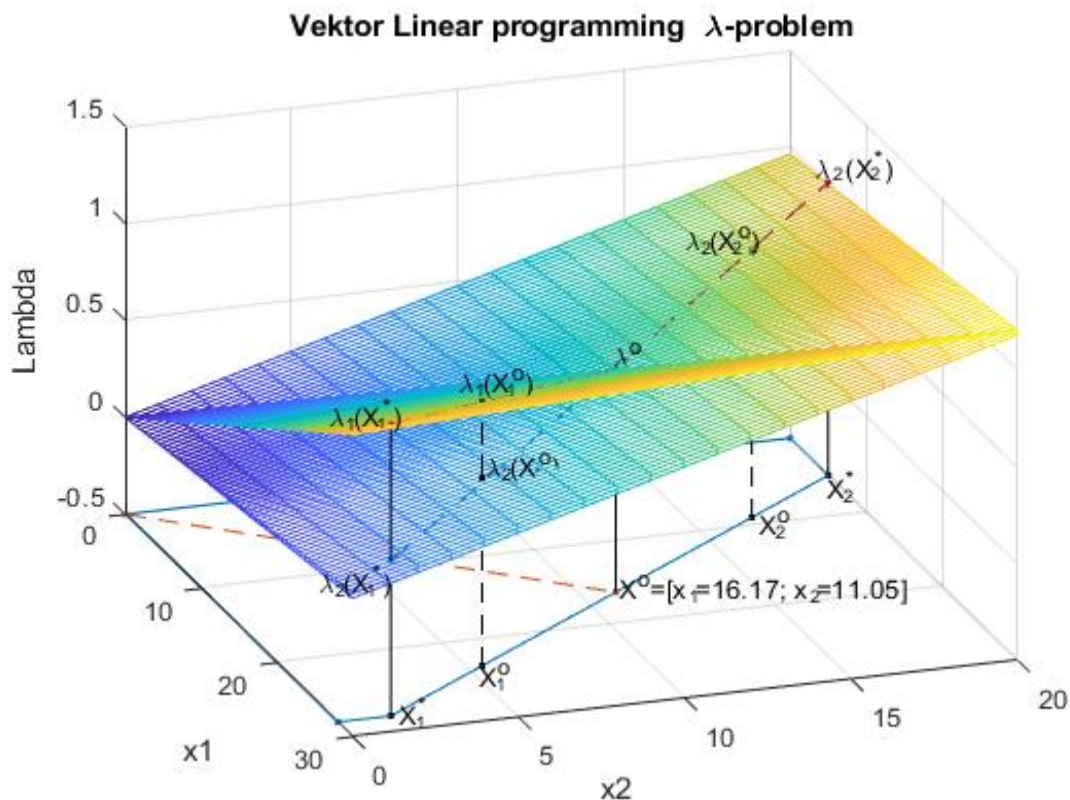


Figure 2. Solution of VPMP (57) and (58) with first-criterion priority

We solve VPMP (57) and, respectively, the λ -problem (59) with the assigned vector of priorities $P^1 = \{p_1^1 = 1, p_2^1 = 2\}$ and the weight vector:

$$w = \{w_1 = 2/3, w_2 = 1/3\}.$$

In this case, the λ -problem takes the form:

$$\begin{aligned} \lambda^o &= \max \lambda, \\ \lambda - p_1^1(5x_1 + 1x_2)/f_1^* &\leq 0, \lambda - p_2^1(20x_1 + 80x_2)/f_2^* \leq 0, \\ x_1 + 5x_2 &\leq 125, x_1 \leq 28, x_2 \leq 20, x_1 \geq 0, x_2 \geq 0, \end{aligned} \quad (60)$$

For this problem, in the transition from the point X^o (obtained with the equivalent criteria of the λ -problem (59)) to the optimal point X_1^* (obtained when solving for the first criterion), the priority of the first criterion lies in the following range: $p_1^1(X) \leq p_2^1 \leq p_2^1(X^o)$.

The concept of the priority of a criterion follows from the axioms, which show that the admissible set of points S (including the set of optimal Pareto points S^o , which lies between the points X_1^* and X_2^*) is divided into three subsets of points:

- the subsets S_1 and S_1^o , which represent the domain of the priority of the first criterion over the second criterion (the domain is characterized by the fact that $\lambda_1(X) \geq \lambda_2(X)$, $\forall X \in S_1^o \subset S_1 \subset S$), during which S_1^o lies between the points X_1^* and X^o ;



- the subsets S_2 and S_2^o , which represent the domain of the priority of the second criterion over the first criterion $\lambda_2(X) \geq \lambda_1(X)$, $\forall X \in S_2^o \subset S_2 \subset S$, during which S_2^o lies between the points X_2^* and X^o ;

- the subset of points $S' \in S$, where the criteria are equivalent $\lambda_1(X) = \lambda_2(X)$, $\forall X \in S' \subset S$. The subset S' contains the point X^o at which λ^o is the maximum level; here, $\lambda^o = \lambda_1(X^o) = \lambda_2(X^o)$ in accordance with the theorem. In this case, X^o also belongs to the set of optimal Pareto points S^o (Figs. 1, 2)

Inequalities (61) in problem (57) take the form $p_1^1(X_1^*) = 3.3245 \leq p_2^1 \leq p_2^1(X^o)$; from there we choose p_2^1 (e.g., $p = 2$), which is introduced into λ -problem (60).

As a result of solving the λ -problem (60) with the assigned vector of priorities, we obtain the following points:

- the optimal point $X_1^o = \{\lambda_1 = 0.8694, x_1 = 23.5959, x_2 = 5.1233\}$ (Fig. 2);
- the optimal value of the goal function $\lambda^o = 0.8694$.

We perform a check: $f_1(X^o) = 123.1$, $\lambda_1(X_1^o) = 0.8694$; $f_2(X_1^o) = 1086.7$, $\lambda_2(X_1^o) = 0.4347$; i.e., $\lambda^o \leq \lambda_1(X_1^o) = 2\lambda_2(X_1^o) = 0.8694$.

These results demonstrate that at the optimal point X_1^o , the relative estimate $\lambda_1(X_1^o)$ is twice as large as the relative estimate $\lambda_2(X_1^o)$. This is in agreement with the formulation of the problem (of the priority of the first criterion over the second criterion); here, the plane determined by the function $2\lambda_2(X)$ in the Pareto domain at the point X_1^o attains the level $\lambda^o = 0.8694 = 2\lambda_2(X_1^o)$. This is seen in the dashed line that passes through the point X_1^o .

As a result of solving the λ -problem (60) with the assigned weight vector, we obtain the following points:

- the optimal point $X_2^o = \{\lambda_1 = 0.2922, x_1 = 9.0615, x_2 = 16.7508\}$ (Figs. 1, 2);
- the optimal value of the goal function: $\lambda^o = 0.2922$.

We perform a check: $f_1(X_2^o) = 62.1$, $\lambda_1(X_2^o) = 0.4383$; $f_2(X_2^o) = 2191.3$, $\lambda_2(X_2^o) = 0.8765$, i.e. $\lambda^o \leq 2\lambda_1(X_2^o)/3 = \lambda_2(X_2^o)/3 = 0.2922$.

These results demonstrate that at the optimal point X_2^o , the relative estimate $\lambda_1(X_2^o)$ is half the relative estimate $\lambda_2(X_2^o)$, and this is in complete contradiction with the initial task.

Thus, the method based on the normalization of criteria and the principle of a guaranteed result, gives accurate answers under the conditions laid down in a VPMP.

Analysis of the results of the solution. (Let's give two explanations).

1. For the first time in domestic and foreign practice, axioms 1, 2 and 2a were constructed, which made it possible to represent in the vector (multi-criteria) problem (1)–(4) (and in particular in the problem II.7- II.8) a permissible set of points S , including:

First, a subset of Pareto-optimal points, $S^o \subset S$, which lies between the optimum points X_k^o , $k = \overline{1, K}$. (In our example, X_k^* , $k = \overline{1, 2} - X_1^*, X_2^*$).

Second, one subset of points $S' \subset S$, where the criteria are equivalent: a subset of points S' , intersecting with a subset of points S^o , separates a subset of Pareto-optimal points into a subset with equivalent criteria $S^{oo} = S' \cap S^o$, which, as shown later (and in our example X^o), consists of a single point $X^o \in S$, i.e. $X^o = S^{oo} = S' \cap S^o$, $S' \in S$, $S^o \subset S$.

Thirdly, " K " subsets of points have been created, where each criterion $q = \overline{1, K}$ has priority over other criteria $k = \overline{1, K}$, $q \neq k$.

As a result, the division is performed:

1) sets of all admissible points S , into subsets of $S_q \subset S$, $q = \overline{1, K}$; In our example, the region of points lying between $X_1^* - 0 - X^o$ is the precedence region of the first criterion $S_{q=1} \subset S$; the region of points lying between $X_2^* - 0 - X^o$ is the priority region of the second criterion $S_{q=2} \subset S$;

2) division of the subset of Pareto-optimal points, $S^o \subset S$, into subsets $S_q^o \subset S_q \subset S$, $q = \overline{1, K}$.

Hence, the following relations are correct:



$$S'U(\bigcup_{q \in K} S_q^o) \equiv S^o, S_q^o \subset S^o \subset S, q = \overline{1, K}.$$

2. We will notice that a subset of points S_q^o is, on the one hand, included in a region (a subset of points) that take precedence over the criterion $q \in K$ over the other criteria: $S_q^o \subset S_q \subset S$, and on the other hand, on a subset of points, Pareto poses are optimal: $S_q^o \subset S^o \subset S$.

Аксиома 2 и числовое выражение приоритета критерия (Определение 5) позволяет определять каждую допустимую точку $X \in S$ (посредством вектора приоритетов:

Axiom 2 and the numerical expression of the criterion's priority (Definition 5) make it possible to define every valid point $X \in S$ by means of the priority vector:

$$P^q(X) = \{p_k^q(X) = \frac{\lambda_q(X)}{\lambda_k(X)}, k = \overline{1, K}\};$$

form and select: a subset of points according to the priority criterion S_q , which is included in the set of points S , $\forall q \in K X \in S_q \subset S$, (such a subset of points can be used in clustering problems, but this is outside the article);

A subset of the points by the priority criterion S_q^o , which is included in the series of Pareto-optimal S^o points, $\forall q \in K, X \in S_q^o \subset S^o$.

Table

Multiple valid points $X \in S$ $\rightarrow$	Subsets of Pareto optimum points, $X \in S^o \subset S \rightarrow$	A subset of points, Pareto optimum $X \in S_q^o \subset S^o \subset S \rightarrow$	A single point, $\forall X \in S$ $X \in S_q^o \subset S^o \subset S$
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This is the most important result that will allow us to derive the principle of optimality and build methods for selecting any point from the set of Pareto-optimal points.

1.5. Applied Multidimensional Mathematics: Vector Problem of Nonlinear Programming as a Model of Engineering Systems Development

In this section, we consider a separate problem of multidimensional mathematics: the vector nonlinear programming problem. Mathematical and software for solving a vector problem of nonlinear programming, the algorithms for solving which with equivalent criteria are presented in Section 3 and with a given priority of the criterion in Section 4, [20, 22, 43, 45].

1.5.1. Vector Problem of Nonlinear Programming

The vector problem of nonlinear programming (VPNP) is a standard problem of mathematical programming, which has some set of criteria that together represent a vector of criteria. VPNP are divided into homogeneous and heterogeneous VPNPs. In accordance with these definitions, we will present a vector problem of nonlinear programming with heterogeneous criteria.

$$Opt F(X) = \{max F_1(X) = \{max f_k(X) = c_{0k} + c_{1k}x_1 + \dots + c_{N+1,k}x_1x_2 + \dots + c_{n1,k}x_1^2 + \dots + c_{nnk}x_n^2, k = \overline{1, K_1}, (61)$$

$$min F_2(X) = \{min f_k(X) = c_{0k} + c_{1k}x_1 + \dots + c_{N+1,k}x_1x_2 + \dots + c_{n1,k}x_1^2 + \dots + c_{nnk}x_n^2, k = \overline{1, K_2}\}, (62)$$

$$a_{0k} + a_{1i}x_1 + \dots + a_{N+1,i}x_1x_2 + \dots + a_{n1,i}x_1^2 + \dots + a_{nni}x_n^2 \leq b_i, i = \overline{1, M}, (63)$$

$$0 \leq x_j(t) \leq u_j, j = \overline{1, N}, (64)$$

where $X = \{x_j, j = \overline{1, N}\}$ is a vector of variables, i.e. it is a vector of the N-dimensional Euclidean space R^N , (the notation $j = \overline{1, N}$ is equivalent to $j = (1, \dots, N)$);

$F(X)$ is a vector function (vector criterion) that has K is a component of functions. The function represents a quadratic polynomial. The set of criteria (polynomials) K consists of a subset of the K_1 component of maximization and a subset of the K_2 minimization: $K = K_1 \cup K_2$, to evaluate the set of



criteria, the designation "opt" operation is introduced, which includes max and min;

$F_1(X) = \{\max f_k(X), k = \overline{1, K_1}\}$ is a vector criterion (51), each component of which maximizes, K_1 is the number of criteria, and the $K_1 = \overline{1, K_1}$ is the set of maximization criteria (task (51), (53)-(54) are VPNNs with homogeneous maximization criteria);

$F_2(X) = \{\min f_k(X), k = \overline{1, K_2}\}$ is a vector criterion, each component of which is minimized, $K_2 = \overline{1, K_2}$ is a set of minimization criteria, K_2 is the number of criteria, (task (52), (53)-(54) is an VPNN with homogeneous minimization criteria). $K_1 \cup K_2 = K$, $K_1 \subset K$, $K_2 \subset K$. (53) - (54) - standard nonlinear constraints (in the form of polynomials, including linear ones). $S = \{X \in R^n | G(X) \leq 0, X^{\min} \leq X \leq X^{\max}\} \neq \emptyset$ is the allowable set of points (or more briefly, the allowable set) given by the standard constraints (53)-(54) and the trivial constraints $0 \leq x_j(t) \leq u_j, j = \overline{1, N}$. Suppose that the allowable set of points is not empty and is compact. The vector function (criterion) of minimization $F_2(X)$ can be transformed into a vector function (criterion) of maximization by multiplying each component of $F_2(X)$ by minus one. Vector criterion $F_2(X)$ is introduced in the VPNN (51)-(54) for in order to show that the problem has two subsets of criteria K_1, K_2 with fundamentally different directions of optimization, [46].

1.5.2. Software Structure for Solving a Vector Problem of Nonlinear Programming

The software for solving the vector problem of nonlinear programming (51)-(54) is implemented on the basis of the algorithm for solving the VPNN and the use of the FMINCON program (...) in the MATLAB system, [20, 21, 22].

When using the FMINCON(...) program, it is necessary to develop two subroutines – functions.

The first function includes two blocks: the first block is designed to evaluate the criterion $f_k(X) \forall k \in K$ at point X ; the second block to calculate the first derivative at this point $\frac{df_k(X)}{dX} \forall k \in K$. The second function includes the same two blocks for constraints only: $g_i(X) \forall i \in M$ and $\frac{dg_i(X)}{dX} \forall i \in M$.

The FMINCON(...) program is used in the first step of the algorithm (maximizing the criteria) and in the second step of the algorithm (minimization). Similarly, in accordance with the algorithm, step 4 and 5 are solved λ -problem.

In general, under nonlinear constraints, the software of the VPNN solution includes:

$K * 2$ (1 step) + $K * 2$ (2 step) + 2 (λ -problem) functions. Since the criteria and limitations of the VPNN are individual, individual software is written for each VPNN.

1.5.3. Example. Numerical implementation of a vector problem of nonlinear programming

We will present the software for the VPNN solution on the example of solving a test vector problem of nonlinear programming.

Example 1.

What is given. The consideration of the vector nonlinear (convex) programming problem with four homogeneous criteria. In terms of criteria, we use a circle, and with the linear restrictions the problem is therefore solved orally and imposed on variables.

$$\text{opt } F(X) = \{\min F_2(X) = \min f_1(X) \equiv (x_1 - 80)^2 + (x_2 - 80)^2, \quad (65)$$

$$\min f_2(X) \equiv (x_1 - 80)^2 + (x_2 - 20)^2, \quad (66)$$

$$\min f_3(X) \equiv (x_1 - 20)^2 + (x_2 - 20)^2, \quad (67)$$

$$\min f_4(X) \equiv (x_1 - 20)^2 + (x_2 - 80)^2, \quad (68)$$

$$\text{at restriction } 0 \leq x_1 \leq 100, 0 \leq x_2 \leq 100, \quad (69)$$



What is required: To find the non-negative solution of x_1, x_2 in the system of inequalities (59), at which the $f_1(X), f_2(X), f_3(X)$ and $f_4(X)$ functions perhaps accept the minimal value.

To solve the problem (55)-(59) for each criterion, as well as the further λ -problem, the MATLAB system (the function `fmincon(...)` — the solution to a non-linear problem of optimization) is used [43]. The solution is presented as a sequence of steps.

Step 1. VPLP (55)-(59) for each criterion is solved (max).

As a result of the decision, we will receive:

Criterion 1: $X_1^* = \{x_1 = 0, x_2 = 0\}, f_1^* = f_1(X_1^*) = -12800$;

Criterion 2: $X_2^* = \{x_1 = 0, x_2 = 100\}, f_2^* = f_2(X_2^*) = -12800$;

Criterion 3: $X_3^* = \{x_1 = 100, x_2 = 100\}, f_3^* = f_3(X_3^*) = -12800$;

Criterion 4: $X_4^* = \{x_1 = 100, x_2 = 0\}, f_4^* = f_4(X_4^*) = -12800$.

Step 1. VPLP (55)-(59) for each criterion is solved (min).

As a result of the decision, we will receive:

Step 2. VPLP (55)-(59) for each criterion is solved (min - opt).

As a result of the decision we will receive:

Criterion 1: $X_1^0 = \{x_1 = 80, x_2 = 80\}, f_1^0 = f_1(X_1^0) = 0$;

Criterion 2: $X_2^0 = \{x_1 = 80, x_2 = 20\}, f_2^0 = f_2(X_2^0) = 0$;

Criterion 3: $X_3^0 = \{x_1 = 20, x_2 = 20\}, f_3^0 = f_3(X_3^0) = 0$;

Criterion 4: $X_4^0 = \{x_1 = 20, x_2 = 80\}, f_4^0 = f_4(X_4^0) = 0$.

Let's present a geometric interpretation of the limitations of the VPMP (55)-(59) and the results of the solution in Figure 3.

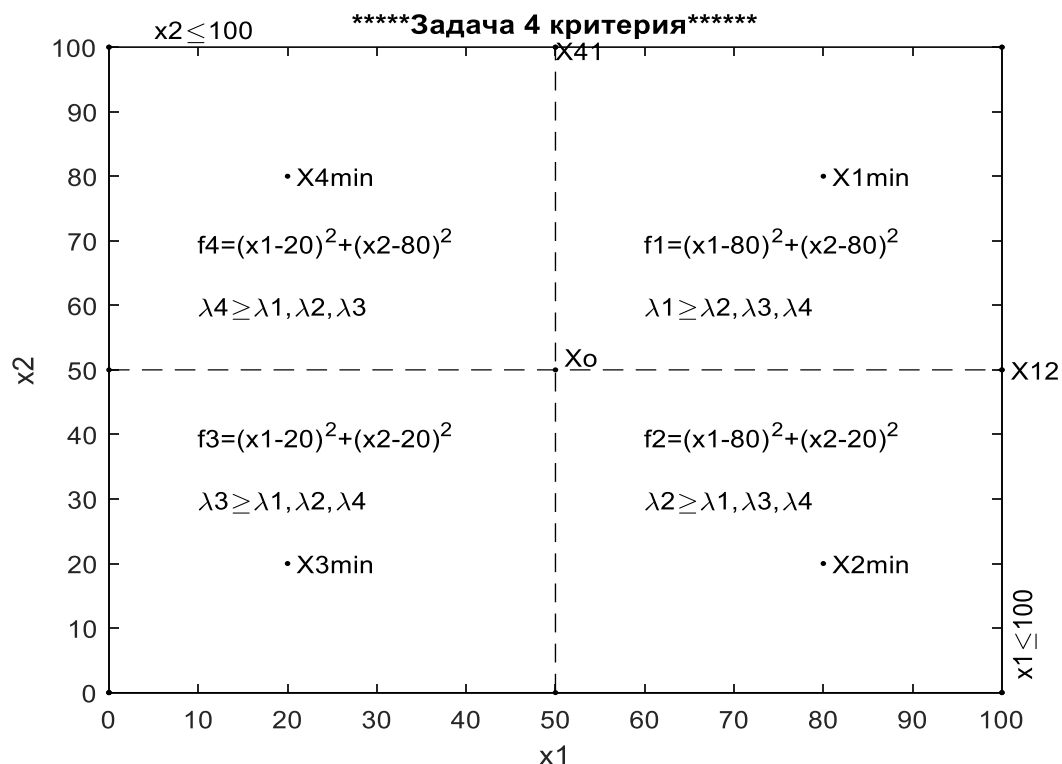


Figure 3. A geometrical interpretation of VPMP (55)-(59). The results of the solution to VPMP

Step 3. A systems analysis of VPLP (55)-(59) criteria is made. For this purpose, at optimum points of $X_1^0 = X1min, X_2^0 = X2min, X_3^0 = X3min$ and $X_4^0 = X4min$ the values of objective



functions $F(X^*)$ and the relative estimates are defined: $\lambda(X^*)$,

$$\lambda_k(X) = \frac{f_k(X) - f_k^0}{f_k^* - f_k^0}, \forall k \in K:$$

$$\begin{aligned} F(X^*) &= \{f_k(X_k^*), q = \overline{1, K}, k = \overline{1, K}, K = 4\} = \\ &= \begin{vmatrix} f_1(X_1^*) & f_2(X_1^*) & f_3(X_1^*) & f_4(X_1^*) \\ f_1(X_2^*) & f_2(X_2^*) & f_3(X_2^*) & f_4(X_2^*) \\ f_1(X_3^*) & f_2(X_3^*) & f_3(X_3^*) & f_4(X_3^*) \\ f_1(X_4^*) & f_2(X_4^*) & f_3(X_4^*) & f_4(X_4^*) \end{vmatrix} = \begin{vmatrix} 0 & 3600 & 7200 & 3600 \\ 3600 & 0 & 3600 & 7200 \\ 7200 & 3600 & 0 & 3600 \\ 3600 & 7200 & 3600 & 0 \end{vmatrix}, \\ \lambda(X^*) &= \begin{vmatrix} \lambda_1(X_1^*) & \lambda_2(X_1^*) & \lambda_3(X_1^*) & \lambda_4(X_1^*) \\ \lambda_1(X_2^*) & \lambda_2(X_2^*) & \lambda_3(X_2^*) & \lambda_4(X_2^*) \\ \lambda_1(X_3^*) & \lambda_2(X_3^*) & \lambda_3(X_3^*) & \lambda_4(X_3^*) \\ \lambda_1(X_4^*) & \lambda_2(X_4^*) & \lambda_3(X_4^*) & \lambda_4(X_4^*) \end{vmatrix} = \begin{vmatrix} 1.0 & 0.7188 & 0.4375 & 0.7188 \\ 0.7188 & 1.0 & 0.7188 & 0.4375 \\ 0.4375 & 0.7188 & 1.0 & 0.7188 \\ 0.7188 & 0.4375 & 0.7188 & 1.0 \end{vmatrix}. \end{aligned}$$

The Pareto set lies between optimum points $X_1^0 = X1min, X_2^0 = X2min, X_3^0 = X3min$ and $X_4^0 = 4min$, i.e., the area of admissible points of S formed by restrictions (59) coincides with a point set, which is Pareto-optimal $S^o, S^o=S$.

At points of an optimum of $X_k^*, k = \overline{1, K}$ all relative estimates (the normalized criteria) are equal to zero: $\lambda_k(X_k^*) = \frac{f_k(X_k^*) - f_k^0}{f_k^* - f_k^0} = 0, k = \overline{1, K}, K = 4$.

At points of an optimum of $X_k^0, k = \overline{1, K}$ (anti optimum) all relative estimates are equal to the unit: $\lambda_k(X_k^0) = \frac{f_k(X_k^0) - f_k^0}{f_k^* - f_k^0} = 1, k = \overline{1, K}, K = 4$. From here, $\forall k \in K, \forall X \in S, 0 \leq \lambda_k(X) \leq 1$.

Step 4. λ -problem is under construction:

$$\begin{aligned} \lambda^o &= \max_{X \in S} \lambda, \\ \text{at restrictions: } \lambda - \frac{f_1(X) - f_1^0}{f_1^* - f_1^0} &\leq 0, \lambda - \frac{f_2(X) - f_2^0}{f_2^* - f_2^0} \leq 0, \\ \lambda - \frac{f_3(X) - f_3^0}{f_3^* - f_3^0} &\leq 0, \lambda - \frac{f_4(X) - f_4^0}{f_4^* - f_4^0} \leq 0, \\ 0 &\leq x_1 \leq 100, 0 \leq x_2 \leq 100 \end{aligned} \quad (70)$$

Step 5. *Solution to the λ -problem.* The solution to the λ -problem in the MATLAB system is similar to step 0 initial parameters

The appeal to the `fmincon(...)` function for the solution to the λ -problem is presented. Results of the solution to the λ -problem:

$$X^o = \{x_1 = 50.0, x_2 = 50.0, x_3 = 0.8594\}.$$

where X^o defines the best values of variables; coordinate of x_1 corresponds to λ^o – maximum relative level: $x_3 = \lambda^o$; and x_1, x_2 corresponds x_1, x_2 to of a problem ((41)-(45).

$Lo = \lambda^o = 0.8594$ represents the optimal value of the objective function.

λ^o is the maximum level among all minimum relative levels on an admissible set of $X \in S$:

$$\lambda^o = \max_{X \in S} \min_{k \in K} \lambda_k(X).$$

The functions $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$, and points of an optimum of X^o and λ^o which are received on their crossing are shown in Figure 2.

As in a task (55)-(59) criteria are symmetric and the point of the guaranteed result of $X^o = \{x_1 = 50., x_2 = 50.\}$ is easily defined – it lies in the center of a square, and a maximal relative assessment of $\lambda^o = 0.5833$. Really, for example, with the first criterion: $\lambda_1(X^o) = \frac{f_1(X^o) - f_1^0}{f_1^* - f_1^0} = ((50 - 80)^2 + (50 - 80)^2 - 12800) / (0 - 12800) = 0.8594$, and similarly for other criteria:



$$\lambda^0 = \lambda_1(X^0) = \lambda_2(X^0) = \lambda_3(X^0) = \lambda_4(X^0) = 0.8594.$$

This is very clearly shown in Figure 4.

In Figures 3 and 4 we can see that the area (point set) limited to function:

$$f_1 = (x_1 - 80)^2 + (x_2 - 80)^2 \text{ is characterized by } \lambda_1(X) \geq \lambda_k(X), k = \overline{2,4}, X \in S_1, \text{ (Figure$$

3 shows how $\lambda_1 > \lambda_2, \lambda_3, \lambda_4$), i.e., it is prioritized by the first criterion. In this area, the priority of the first criterion is always more or equal to the unit: $p_k^1(X) = \lambda_1(X)/\lambda_k(X) \geq 1, \forall X \in S_1$.

Areas (point set) prioritized by the corresponding criterion are similarly shown. In total, they give a point set, which is Pareto-optimal, of S^0 , and it (for this example) is equal to a set of admissible points: $S^0 = S_1^0 \cup S_2^0 \cup S_3^0 \cup S_4^0 \cup X^0 = S$.

1.5.4. Study of the Results of Solving a Vector Problem of Nonlinear Programming as a Mathematical Model of an Engineering System

By the concept of an engineering system, we understand: technical systems, technological processes, material structure. A vector problem of nonlinear (convex) programming with four homogeneous criteria (65)-(69) and the corresponding λ -problem (60) are considered as a mathematical model of an engineering system. Functions in relative units $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$ as well as the optimum point X^0 and λ^0 , which are obtained at the intersection, in the three-dimensional coordinate system x_1, x_2, λ are shown in Fig. 4.

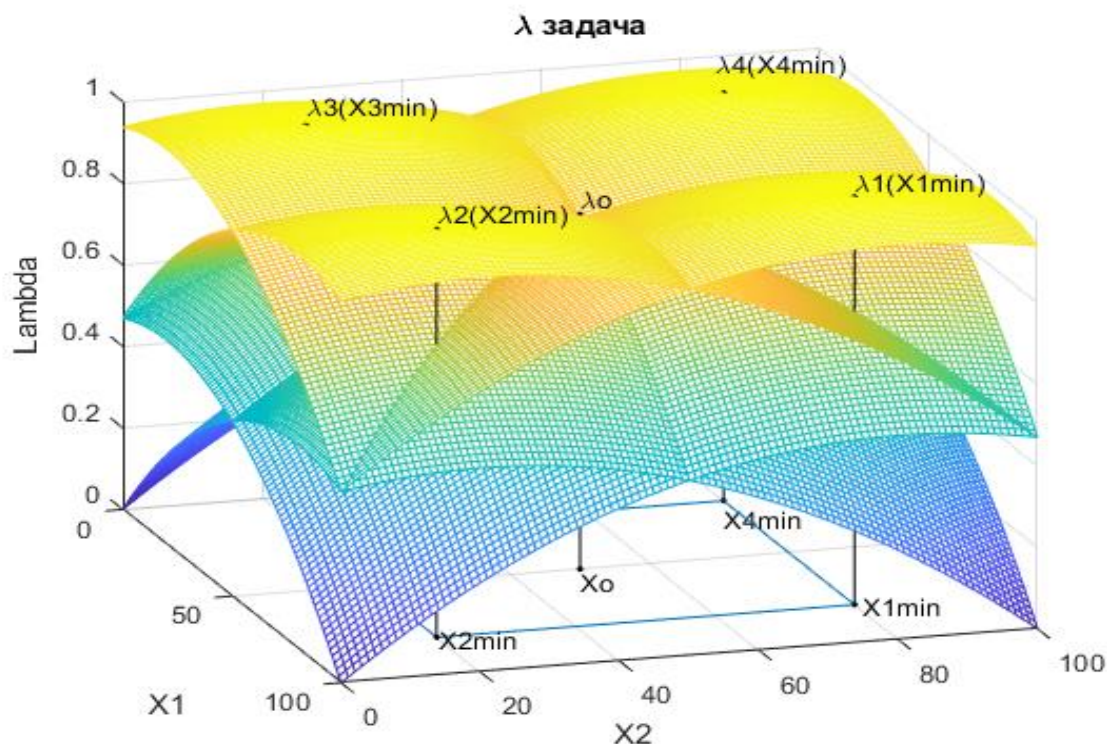


Figure 4. Results of the solution to VPLP (65)-(69) and (70): functions: $\lambda_1(X), \lambda_2(X), \lambda_3(X)$, and $\lambda_4(X)$, points of an optimum of X^0 and λ^0

Let us imagine a solution with equivalent criteria in relative units - λ -problems. As a result of the solution with equivalent criteria (-the problems received the optimum point:



$X^0 = \{x_1 = 50.0, x_2 = 50.0, x_3 = 0.8594\}$, where $x_3 = \lambda^0 = 0.8594$.

$\lambda^0 = 0.8594$ is a guaranteed result in relative units, which shows that an improvement in any criterion leads to a deterioration in other criteria.

Relative units are easily converted into physical units for each criterion (characteristic of the engineering system). The third step is to solve the VPMP with equivalent criteria at the optimum points $X_1^*, X_2^*, X_3^*, X_4^*$. (in Fig. 2: $X1min, X2min, X3min, X4min$) the values of all relative estimates are obtained:

$$L(Y^*) = \begin{Bmatrix} L(X_1^*) \\ L(X_2^*) \\ L(X_3^*) \\ L(X_4^*) \end{Bmatrix} = \begin{Bmatrix} 1.0 & 0.7188 & 0.4375 & 0.7188 \\ 0.7188 & 1.0 & 0.7188 & 0.4375 \\ 0.4375 & 0.7188 & 1.0 & 0.7188 \\ 0.7188 & 0.4375 & 0.7188 & 1.0 \end{Bmatrix}.$$

Let's imagine from Fig. 2 values of relative estimates at the optimum point, for example X_2^* :

$L(X_2^*) = \{\lambda_1(X_2^*) = 0.7188, \lambda_2(X_2^*) = 1.0, \lambda_3(X_2^*) = 0.7188, \lambda_4(X_2^*) = 0.4375\}$.

A linear function connecting the points $\lambda^0 = 0.8594$ and $\lambda_2(X_2^*) = 1$ in relative units characterizes the function $f_2(X)$ in relative units in the two-dimensional dimension of the parameters x_1, x_2 . This linear function represents a geometric interpolation of the functions $f_2(X)$ in relative units from N-dimensional (in this example, 2-dimensional) to a two-dimensional coordinate system.

Suppose that for the designer, the function $f_2(X)$ is the most important-priority. In the study of the priority function $f_2(X)$ we can set different numeric values to $f_2(X)$ from the ratio: $f_2(X^0) \leq f_2(X) \leq f_2(X_2^*)$ and as a result we can obtain the corresponding parameters.

In the same way, you can study other criteria (functions of an engineering system): $f_1(X), f_3(X), f_4(X)$.. The numerical solution of such a study is presented on a four-dimensional system.

Modeling, simulation and optimal decision making based on a numerical set of criteria: a technological process with two parameters.

In this section, we consider a separate problem of multidimensional mathematics: the vector nonlinear programming problem. Mathematical and software for solving a vector problem of nonlinear programming, the algorithms for solving which with equivalent criteria are presented in Section 3 and with a given priority of the criterion in Section 4, [20, 22, 43, 45].

1.5.5. Methodology for choosing the optimal parameters of complex engineering systems under conditions of certainty, uncertainty with equivalent criteria and with a given priority of the criterion, using the intellectual knowledge of the system designer

As an object of research, we consider "Engineering systems", which include "technical systems", "technological processes", "materials", [17, 18 and 19]. The study of the engineering system is carried out, firstly, under conditions of certainty, when the data on the functional characteristics of the engineering system are known; secondly, in conditions of uncertainty, when, using the intellectual knowledge of the designer, discrete values of individual characteristics of the system are set. There is data on the restrictions that are imposed on the functioning of the system. Mathematical apparatus of engineering system modeling is based on theory and methods of vector optimization, which are presented in the second section. In organizational terms, the process of modeling and simulation of the technical system is presented in the form of a methodology:

"Methodology of selection of optimal parameters of engineering systems in conditions of certainty and uncertainty."

1. Types of Problems Arising in the Process of Modeling and Making an Optimal Decision on the Selection of Parameters of Complex Engineering Systems

The problems that arise in the process of making an optimal decision on the selection of optimal parameters of complex Engineering systems on the basis of vector optimization include three types sequentially.



1 type. *Solution of a vector problem of mathematical programming with equivalent criteria.* The result obtained is the basis for further research of the system. In this case, the method of solving a vector problem with equivalent criteria is used. If the result obtained satisfies the decision-maker (decision-maker - designer), then it is taken as a basis. If it does not satisfy, then move on to the second type (direct problem) or the third type of solving vector problems (Inverse problem).

2 type. *The solution of the direct problem of vector optimization*, which consists in the following: "What will be the indicators (characteristics) if the parameters of complex engineering systems are changed." - The method of solving a vector problem with equivalent criteria is used.

3 type. The solution of the inverse problem of vector optimization, which consists in the following: "What will be the parameters of complex engineering systems with the given numerical values of individual characteristics". These characteristics are set using the intellectual knowledge of the designer of the engineering system. A method for solving a vector problem is used for a given priority of the criterion

2. Methodology for Modeling and Making an Optimal Decision on the Choice of Parameters of Complex Engineering Systems in Conditions of Certainty, Uncertainty

The methodology for selecting the optimal parameters of complex engineering systems is divided into two parts. *The first part* of the methodology is intended for "Selection of optimal parameters of engineering systems in conditions of certainty and uncertainty with equivalent criteria. The second part of the methodology is intended for "Selection of optimal parameters of engineering systems in conditions of certainty and uncertainty with the priority of one of the criteria". The numerical values of the criterion under study are set using intellectual knowledge by the designer of the engineering system. The first and second parts include two blocks, divided into a number of stages.

Part 1. Selection of optimal parameters of complex engineering systems under conditions of certainty and uncertainty with equivalent criteria.

Block 1. *The formation of technical specifications*, the transformation of uncertainty conditions (related to experimental data) into conditions of certainty, the construction of a mathematical and numerical model of an engineering system (the process of modeling of an engineering system) includes 4 stages.

Stage 1. Formation of technical specifications (initial data) for numerical modeling and selection of optimal system parameters. The initial data is formed by the designer who designs the engineering system.

Stage 2. Construction of mathematical and numerical model of the engineering system in conditions of certainty and uncertainty.

Stage 3. Transformation of uncertainty conditions into certainty conditions and construction of a mathematical and numerical model of an engineering system under conditions of certainty.

Stage 4. Construction of an aggregated mathematical and numerical model of an engineering system under conditions of certainty

Block 2. *Making an optimal decision* (selection of optimal parameters) in an engineering system with equivalent criteria based on multidimensional mathematics.

Stage 5. Solution of a vector problem of mathematical programming (VPMP) - a model of an engineering system with equivalent criteria (solution of a direct problem).

Stage 6. Geometric interpretation of the results of the vector problem of mathematical programming solution with N parameters and K criteria into a two-dimensional coordinate system in relative units.

Stage 7. Presentation of the results of solving the Vector Problem of Mathematical Programming with equivalent criteria and preparation of information for decision-making with the priority of a separate criterion from the entire set of criteria.

Part 2. Selection of optimal parameters of engineering systems under uncertainty with the priority of the criterion, the numerical values of which are set by the system designer.



Block 3. Analysis, comparison of the results of the VPMP solution with equivalent criteria of a complex engineering system and preparation for the selection of optimal parameters according to the priority criterion.

Stage 8. Analysis and comparison of the results of solving a vector problem with equivalent criteria with four and two variables. Conclusion from the analysis: *Information about the Engineering System in physical units* for decision-making with the priority of the criterion. Preparation for geometric interpretation of the solution of the vector problem of choosing optimal parameters according to the priority criterion.

Stage 9. Geometric interpretation of the solution results in the design of an engineering system for the transition from two-dimensional to N -dimensional space in relative units.

Block 4. Selection of optimal parameters of a complex engineering system according to a priority criterion, geometric interpretation of the solution results based on multidimensional optimization.

The selection of the optimal parameters of a complex engineering (technical) system and the geometric interpretation of the solution results according to the priority criterion corresponds to the number of criteria $K=4$: Block k_1 , Block k_2 , Block k_3 , Block k_4 .

Name of the block: Block k_1 , ..., Block k_4 . «*Selection of optimal parameters of the engineering system (technical system) according to the (first, ..., fourth) priority criterion, geometric interpretation of the results of the VPMP solution*».

Each parameter selection block for the corresponding criterion includes the following sections.

1.k1, ..., 1.k4. *Solution of a vector problem* - a model of a complex engineering system (technical system) with a given priority of the first criterion, including:

Solving a vector problem of mathematical programming - a model of an engineering system with equivalent criteria (Step 1).

Select the priority criterion $q \in K$ (Step 2).

Select the numerical value of the selected priority criterion $q \in K$ (Steps 3, 4).

Solutions to a vector problem of mathematical programming - a model of an engineering system with a given priority of the $q \in K$ criterion (Steps 5–8).

2.k1, ..., 2.k4. Analysis of the results of solving a vector problem of mathematical programming - a model of an engineering system at a given priority of the first criterion.

3.k1, ..., 3.k4. Geometric interpretation of the results of solving the vector problem of choosing optimal parameters according to the first priority criterion in relative units.

4.k1, ..., 4.k4. Geometric interpretation of the results of the VPMP solution with the priority of the first criterion – the model of the engineering system when designing in a three-dimensional coordinate system *in physical units*.

The other three blocks are formed by analogy.

1. Geometric interpretation of the results of the solution of the first criterion with the priority of the first criterion – the model of the engineering system when designing in a three-dimensional coordinate system *in physical units*. ...

4. Geometric interpretation of the results of the solution of the VZMP of the fourth criterion with the priority of the first criterion - the model of the engineering system when designing in a three-dimensional coordinate system *in physical units*.

2. Modeling and selection of optimal parameters of a technological system under conditions of certainty and uncertainty based on multidimensional mathematics and designer's intelligence.

The numerical implementation of the choice of optimal parameters of the technological system is carried out on the basis of the theory of methods of multidimensional mathematics and the intelligence of the designer.



2.1. Mathematical modeling of the Multiparameter and Multifunctional technological process for making an optimal management decision

2.1.1. Mathematical model of a technological process in conditions of certainty

Please As an object of research, we use "technological process". The problem of decision-making in technology in the production of products is formulated in accordance with the works [15].

We consider a technological process (e.g., Hybrid laser arc welding, [15], in which ze41-T5 alloy was chosen as the material to be welded with AZ61 alloy as the filler material).

Activity of technological process depends on a particular set of conditions - design data:

$$X = \{x_1, x_2, \dots, x_N\}^T, \text{ or } X = \{x_j, j = \overline{1, N}\},$$

(for example: laser powers; speeds of movement; feed rates of a wire; current; frequencies) we Will designate. Let's denote N -set of constructive parameters. Each parameter of the technological process lies in the given limits:

$$x_j^{\min} \leq x_j \leq x_j^{\max}, j = \overline{1, N}, \text{ or } X^{\min} \leq X \leq X^{\max},$$

where $x_j^{\min}, x_j^{\max}, \forall j \in N$ - the lower and top limits of change of a vector of parameters of technological process, N is the number of parameters.

The result of functioning is defined by a set of running characteristics $f_k(X), k = \overline{1, K}$, which functionally depend on design data of technological process of $X = \{x_j, j = \overline{1, N}\}$, (for example: the depth of the weld; underfill; percentage defect; total accumulated pore length). In total all running characteristics represent a vector function:

$$F(X) = (f_1(X) f_2(X) \dots f_K(X))^T, \text{ or } F(X) = \{f_k(X), k = \overline{1, K}\}^T,$$

where K – a set (number) of technological characteristics (criteria).

The set of characteristics K is subdivided into two subsets of K_1 and K_2 :

$$K = K_1 \cup K_2, K_1 \subset K, K_2 \subset K.$$

K_1 is a subset of technological characteristics, the numerical values of which are desirable to obtain as high as possible: $f_k \rightarrow \max, k = \overline{1, K_1}$.

K_2 are subsets of technological characteristics, the numerical values of which are desirable to obtain as low as possible: $f_k \rightarrow \min, k = \overline{1, K_1 + 1}, K_2 \equiv \overline{K_1 + 1, K}$.

Mathematical model has to reflect, first, the purposes of the technological process which are presented by the characteristics of $F(X)$, and secondly, to take into account the constraints of $X^{\min} \leq X \leq X^{\max}$.

The mathematical model of the technological process solving in general a problem of the choice of optimum parameters of the technological process can be presented in the form of a vector problem of mathematical programming.

$$\begin{aligned} \text{Opt } F(X) &= \{\max F_1(X) = \{\max f_k(X), k = \overline{1, K_1}\}, \\ &\min F_2(X) = \{\min f_k(X), k = \overline{1, K_2}\}\}, \end{aligned} \quad (71)$$

$$\begin{aligned} \text{the restrictions: } G(X) &\leq 0, f_k^{\min} \leq f_k(X) \leq f_k^{\max}, k = \overline{1, K} \\ x_j^{\min} &\leq x_j \leq x_j^{\max}, j = \overline{1, N}, \end{aligned} \quad (72)$$

where X is the vector of operated variable (design parameters) of the technological process;

$F(X) = \{f_k(X), k = \overline{1, K}\}$ – vector criterion which each component represents the characteristic of the technological system (71) which is functionally depending on a vector of variables X ; in (72) $G(X) = \{g_1(X) g_2(X) \dots g_M(X)\}^T$ is a vector function of the restrictions imposed on functioning of the technological process, M is a set of constraints. The constraints are defined flowing past in them technological, physical and to that similar processes and can be presented by the functional restrictions, for example: $f_k^{\min} \leq f_k(X) \leq f_k^{\max}, k = \overline{1, K}$.

It is supposed that the functions $f_k(X), k = \overline{1, K}$ are differentiated and convex, $g_i(X), i = \overline{1, M}$ are continuous, and $S = \{X \in R^n | G(X) \leq 0, X^{\min} \leq X \leq X^{\max}\} \neq \emptyset$.



The (71)-(72) ratios form mathematical model of the technological process. It is required to find such vector of the $X^0 \in S$ parameters at which each component of a vector function of

$F_1(X) = \{f_k(X), k = \overline{1, K_1}\}$ accepts the greatest possible value, and vector functions of $F_2(X) = \{f_k(X), k = \overline{1, K_2}\}$ accepts minimum value.

2.1.2. Mathematical Model of Multiparameter and Multifunctional Technological Process under Uncertainty

In the vector problem of mathematical programming (71)-(72), the functional dependence of each criterion (71) and constraints (72) on the parameters imposed on the functioning is known. Such a mathematical model (71)-(72) is called a model under *conditions of certainty*.

If there is not enough information about the functional dependence of each criterion (71) and constraints (72) on the parameters, and only discrete (usually experimental) data are known, $I_i(X)$, then such a model of an engineering system is characterized as a model *under uncertainty*.

Let's imagine a discrete set of parameters and characteristics:

$$I_i(X) = \{X_i, f_k(X_i), i = \overline{1, M}\}^T = \begin{bmatrix} x_{i1}, \dots, x_{iN} & f_{i1}(X_i), \dots, f_{iK}(X_i) \\ \dots & \dots \\ x_{i1}, \dots, x_{iN} & f_{i1}(X_i), \dots, f_{iK}(X_i) \end{bmatrix} \quad \text{in conditions of}$$

uncertainty, where N is a set of parameters, M is a set of data, or $X_i = \{x_{ij}, j = \overline{1, N}, i = \overline{1, M}\}$.

To obtain data $I_i(X) = \{(y_k(X_i), i = \overline{1, M})\}^T$ experimental studies of the technological system are carried out on the principle of "input-output", while the problem of decision-making under conditions of uncertainty is formed and the problem of choosing the optimal estimate based on the data obtained arises.

2.1.3. Mathematical Model of Multiparameter and Multifunctional Technological Process in Conditions of Certainty and Uncertainty in the Aggregate

The mathematical model of the technological process (71) - (72) is made under conditions of certainty. In real life, the conditions of certainty and uncertainty are combined. The model of the technological process should also reflect these conditions. Using the designations of the mathematical model, we transform the model (71) - (72) taking into account the conditions of uncertainty, as a result, we get:

"Model of the technological system in conditions of certainty and uncertainty":

$$\text{Opt } F(X) = \{\max F_1(X) = \{\max f_k(X), k = \overline{1, K_1^{def}}\}, \quad (73)$$

$$\max I_1(X) \equiv \{\max f_k(X_i, i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}, \quad (74)$$

$$\min F_2(X) = \{\min f_k(X), k = \overline{1, K_2^{def}}\}, \quad (75)$$

$$\min I_2(X) \equiv \{\min f_k(X_i, i = \overline{1, M})\}^T, k = \overline{1, K_2^{unc}} \quad (76)$$

$$\text{the restrictions: } f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1, K},$$

$$G(X) \leq 0, x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1, N}, \quad (77)$$

where $X = \{x_j, j = \overline{1, N}\}$ - vector of controlled variables (parameters) of the technological system;

$F(X) = \{F_1(X) F_2(X) I_1(X) I_2(X)\}$ - a vector criterion, each component of which represents a vector of criteria (characteristics) of the technological process (73) - (74), which functionally depend on the discrete values of the variable vector, where X in (73) and (74) K_1^{def}, K_2^{def} (*definiteness*), а в (73) и (74) K_1^{unc}, K_2^{unc} (*uncertainty*) a set of criteria max and min formed in conditions of certainty and certainty.

The criteria $I_1(X) I_2(X)$ characterize discrete values of functions that are functionally dependent on the discrete values of the vector of variables X , denoted in (73) and (74) as K_1^{unc}, K_2^{unc} (*uncertainty*) set of max and min criteria. Formed in conditions of uncertainty.

(75) and (76) standard constraints: functional, parametric and general.



2.2. Part 1. Selection of optimal parameters of the technological process under conditions of certainty and uncertainty with equivalent criteria.

The terms of reference, the analysis of the results of the solution and the choice of the priority criterion, its magnitude are carried out by the material *designer*. The remaining stages are performed by a *mathematician-programmer*.

2.2.1. The technical assignment: The choice of the optimal parameters of the technological process

It is given. The technological process, the operation of which is determined by two parameters $X = \{x_1, x_2\}$ - the vector of variables (controlled): $2.0 \leq x_1 \leq 3.5$, $12.0 \leq x_2 \leq 30.0$.

The operation of the process is determined by four characteristics (criteria):

$F(X) = \{f_1(X), f_2(X), f_3(X), f_4(X)\}^T$, the value of which depends on the vector of parameters X .

Conditions of certainty. The conditions are characterized by the fact that the functional dependence of the fourth characteristic $f_4(X)$ on the parameters of the technological process $X = \{x_1, x_2\}$ is known:

$$f_4(X) = -0.2450 - 0.7470x_1 + 0.3832x_1^2 + 0.0442x_2 + 0.0012x_2^2 - 0.0346x_1x_2. \quad (77)$$

Conditions of uncertainty. For the first, second and third characteristics of the technological process, the results of experimental data are known: the values of the parameters and the corresponding characteristics. The numerical values of the parameters $X = \{x_1, x_2\}$ and the characteristics $f_1(X), f_2(X), f_3(X)$ are presented in Table 2.

Table 2

Experimental weld input and output parameters

Laser Power, p (Analog V)	Travel Speed, v (mm/sec)	Wire Feed Rate, r (m/min)	Depth, D(mm)	Total Accumulated Pore Length, Po (mm/mm)
x_1	x_2	$y_1(X) \rightarrow \max$	$y_2(X) \rightarrow \min$	$y_3(X) \rightarrow \max$
2.401	24.2	4.2197	55.3951	-0.0365
2.760	18.72	3.2714	31.2497	0.0286
2.761	19.08	3.2770	32.3886	0.0271
2.760	31.68	4.2613	86.8526	0.0760
2.761	31.92	4.2949	88.1656	0.0787
3.300	14.40	3.0959	21.1331	0.3467
3.301	25.20	3.0101	56.1913	0.2171
3.300	25.80	3.0383	58.7506	0.2138
3.301	26.76	3.0908	62.9794	0.2096
3.300	27.60	3.1441	66.8147	0.2068
3.301	28.80	3.2320	72.5126	0.2041
3.300	30.00	3.3338	78.4682	0.2032
3.301	31.20	3.4495	84.6812	0.2039
3.300	32.40	3.5792	91.1518	0.2063
3.301	36.00	4.0517	112.1086	0.2236
3.840	18.72	3.0983	35.3082	0.6402
3.841	23.52	2.9671	51.6458	0.5450
3.840	31.68	3.2554	88.8758	0.4451
3.840	32.88	3.3520	95.3551	0.4369
4.201	25.20	3.2370	60.4633	0.7810
Max: 2.40	14.4	2.9671	21.1331	-0.0365
Max: 4.20	36.0	4.2949	112.1086	0.7810



In the decision taken, the evaluation value for the first, second and third characteristics (criteria) are desirable to get as high as possible: $y_1(X) \rightarrow \mathbf{max}$, $y_2(X) \rightarrow \mathbf{min}$, $y_3(X) \rightarrow \mathbf{max}$, $f_4(X) \rightarrow \mathbf{min}$. The parameters $X = \{x_1, x_2\}$ vary within the following limits:

$$2 \leq x_1 \leq 3.5, 12 \leq x_2 \leq 30. \quad (78)$$

It is required. Build a model of the technological process in the form of a vector problem. Solve the vector problem with equivalent criteria. Choose a priority criterion. Set the numerical value of the priority criterion. Solve the vector optimization problem and make the best (optimal) solution with the given criterion priority.

2.2.2. Construction of a mathematical (numerical) model of a technological process under conditions of certainty

The construction of a model of a technological process with functional dependence is determined by the fact that the characteristics and limitations of the parameters $X = \{x_1, x_2\}$ are known. Characteristics (77) and limitations (78) are known. Using data (77), (78), we construct a one-criteria problem of nonlinear programming - conditions of certainty:

$$f_4(X) = -0.245 - 0.747x_1 + 0.3832x_1^2 + 0.0442x_2 + 0.0012x_2^2 - 0.0346x_1 * x_2, \quad (79)$$

$$2.0 \leq x_1 \leq 3.5, 12.0 \leq x_2 \leq 30.0. \quad (80)$$

In the future, information data (79) and (80) are used in the construction of an aggregated mathematical technological process.

2.2.3. Transformation of uncertainty conditions (experimental data) into conditions of certainty and construction of a numerical model

Digital transformation consists in the use of qualitative and quantitative descriptions of the technological system obtained on the principle of "input-output" in Table 2. Converting information (initial data $f_1(X), f_2(X), f_3(X)$) into a functional form $f_1(X) f_2(X) f_3(X)$ is carried out by using mathematical methods (the regression analysis). The initial data of table 2 are formed as a matrix I in the MATLAB system:

$$I = |X, F| = \begin{bmatrix} X_1 = \{x_{11}, x_{12}\}, f_1(X_1), f_2(X_1), f_3(X_1) \\ \dots \\ X_M = \{x_{M1}, x_{M2}\}, f_1(X_M), f_2(X_M), f_3(X_M) \end{bmatrix}. \quad (81)$$

For each experimental data set $f_k, k = \overline{1,3}$ the regression function is constructed using the least squares method $\min \sum_{i=1}^M (f_i - \bar{f}_i)^2$ in the MATLAB system. For this, the polynomial A_k is formed, which determines the interrelation of the parameters $X_i = \{x_{i1}, x_{i2}\}$ and the function

$$\bar{f}_{ki} = f(X_i, A_k), X_{ki} = \{x_{ki}, x_{ki}\}, k = \overline{1,3}. \quad (82)$$

The result is a system of coefficients: $A_k = \{A_{0k}, A_{1k}, \dots, A_{9k}\}$, which determine the coefficients of the polynomial (function):

$$f_k(X, A) = a_{0k} + a_{1k}x_1 + a_{2k}x_1^2 + a_{3k}x_2 + a_{4k}x_2^2 + a_{5k}x_1 * x_2, k = \overline{1,3}. \quad (83)$$

To determine the polynomial coefficients (83) of the functions in Table 2, polynomial approximation software with two variables and five factors is developed.

As a result of the program's work, the coefficients were obtained:

$$Ao = \begin{bmatrix} 11.4751 & 8.8173 & -0.1222 & \% A_{0k} \\ -4.8994 & -7.6807 & -0.3735 & \% A_{1k} \\ 0.8868 & 2.1456 & 0.1916 & \% A_{2k} \\ -0.0030 & 0.1851 & 0.0221 & \% A_{3k} \\ 0.0048 & 0.0894 & 0.0006 & \% A_{4k} \\ -0.0595 & -0.1454 & -0.0173 & \% A_{5k} \end{bmatrix}. \quad (84)$$

As a result, $\{x_{1i}, x_{2i}, f_{1i}, i = \overline{1, M}\}$ - the experimental data of the matrix $I = |X, F|$ (74) are transformed into the function $f_1(X)$ in the form (83), which, taking into account the obtained coefficients, will take the form:



$$f_1(X) = 11.474 - 4.8992x_1 + 0.8868x_1^2 - 0.0030x_2 + 0.0048x_2^2 - 0.0595x_1x_2.$$

The experimental data $I_2 = \{x_{i1}, x_{i2}, f_{i2}\}$ of the matrix $I = |X, Y|$ (84) are converted into the function $f_2(X)$ in the form (83), which, taking into account the obtained coefficients, takes the form: $f_2(X) = 8.817 - 7.6809x_1 + 2.1456x_1^2 + 0.1851x_2 + 0.0894x_2^2 - 0.1454x_1x_2$.

The experimental data $I_3 = \{x_{i1}, x_{i2}, f_{i3}\}$ of the matrix $I = |X, F|$ (84) are converted to the function $f_3(X)$ in the form (83), which, taking into account the obtained coefficients, takes the form: $f_3(X) = -0.1225 - 0.3735x_1 + 0.1916x_1^2 + 0.0221x_2 + 0.0006x_2^2 - 0.0173x_1x_2$.

As a result, the experimental data of Table 1 are formed taking into account the purposefulness of the vector problem of mathematical programming:

$$\max f_1(X) = 11.474 - 4.899x_1 + 0.8868x_1^2 - 0.003x_2 + 0.0048x_2^2 - 0.0595x_1x_2. \quad (85)$$

$$\max f_2(X) = 8.817 - 7.681x_1 + 2.145x_1^2 + 0.1851x_2 + 0.0894x_2^2 - 0.1454x_1x_2. \quad (86)$$

$$\max f_3(X) = -0.1225 - 0.3736x_1 + 0.1916x_1^2 + 0.0222x_2 + 0.0006x_2^2 - 0.0173x_1x_2 \quad (87)$$

$$2.0 \leq x_1 \leq 3.5, 12.0 \leq x_2 \leq 30.0.$$

2.2.4. Mathematical numerical model of the technological process with aggregated experimental data under conditions of certainty

(General part for the conditions of certainty and uncertainty).

To build a mathematical model of a technological process, we use: the functions obtained by the conditions of certainty and uncertainty (79), (85)-(87); parametrical restrictions (87).

Functions (79), (85)-(87) are considered as criteria determining the purposefulness of the operation of the technological process. All criteria are aimed $K_1 = 2$ at maximization: $f_1(X), f_3(X) \rightarrow \max$ and $K_2 = 2$ at minimization: $f_2(X), f_4(X) \rightarrow \min$ $K = K_1 \cup K_2$. As a result, the model of the functioning of the technological process will be represented by the vector problem of mathematical programming:

$$\text{Opt } F(X) = \{\max F_1(X) = \{ \max f_1(X) \equiv 11.474 - 4.899x_1 + 0.8867x_1^2 - 0.0031x_2 + 0.0048x_2^2 - 0.0595x_1x_2, \quad (88)$$

$$\max f_3(X) \equiv -0.1225 - 0.3735x_1 + 0.1916x_1^2 + 0.0221x_2 + 0.0006x_2^2 - 0.0173x_1x_2, \quad (89)$$

$$\min F_2(X) = \{\min f_2(X) \equiv 8.81 - 7.68x_1 + 2.145x_1^2 + 0.185x_2 + 0.089x_2^2 - 0.145x_1x_2, \quad (90)$$

$$\min f_4(X) \equiv -0.2451 - 0.747x_1 + 0.3832x_1^2 + 0.0442x_2 + 0.0012x_2^2 - 0.0346x_1x_2\}, \quad (91)$$

$$\text{restrictions: } 2.01 \leq x_1 \leq 3.5, 12.01 \leq x_2 \leq 30.01. \quad (92)$$

The vector problem of mathematical programming (88)-(92) represents a mathematical model of making optimal decisions in conditions of certainty and uncertainty in the aggregate, as a result of digital transformation.

2.2.5. Optimal Decision Making in a Two-Parameter Four-Dimensional Process Model with Equivalent Criteria.

To solve the vector problem of mathematical programming (88)-(92), we use the solution method based on the normalization of criteria and the principle of guaranteed results, presented in section 4. The solution of the vector problem (88)-(92) with equivalent criteria is represented as a sequence of steps.

Step 1. Solving VPMP (88)-(92) for each criterion separately, using the `fmincon(...)` function of the MATLAB system [19], the call to the `fmincon(...)` function is considered in [15]. As a result of the calculation for each criterion, we obtain optimum points: X_k^* and $f_k^* = f_k(X_k^*)$, $k = \overline{1, K}$, is the values of the criteria at this point, that is, the best solution for each criterion:

$$\begin{aligned} X_1^* &= \{x_1 = 2, x_2 = 30\}, f_1^* = f_1(X_1^*) = -5.8833; \\ X_2^* &= \{x_1 = 3.5, x_2 = 30\}, f_2^* = f_2(X_2^*) = -78.9641; \end{aligned} \quad (93)$$



$$X_3^* = \{x_1 = 3.5, x_2 = 12.0\}, f_3^* = f_3(X_3^*) = -0.54235;$$

$$X_4^* = \{x_1 = 2.9, x_2 = 12.0\}, f_4^* = f_4(X_4^*) = -0.3334.$$

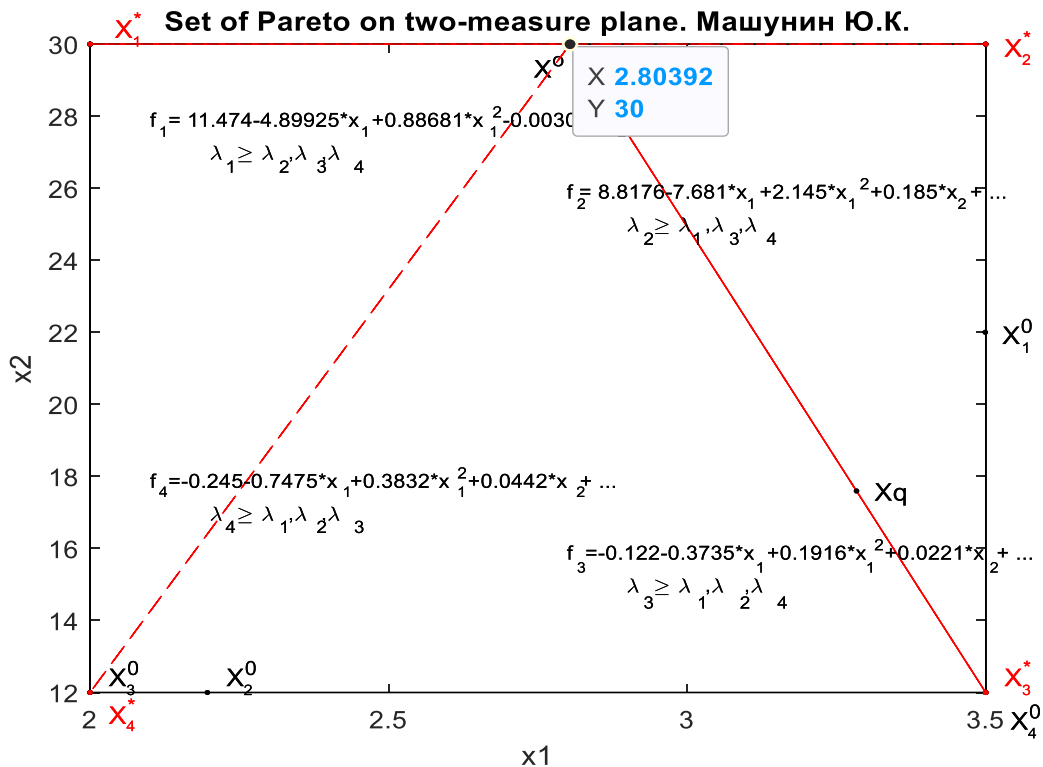


Figure 3. Pareto set, $S^0 \subset S$ in two-dimensional coordinate system, the points of the optimum X_1^* , X_2^* , X_3^* , X_4^*

The constraints (92) and optimum points X_1^* , X_2^* , X_3^* , X_4^* (93) in the coordinates $\{x_1, x_2\}$ are shown in Figure 3. The set of valid points S is not empty and is a compact: $S = \{X \in \mathbf{R}^N \mid 2 \leq x_1 \leq 3.5, 12 \leq x_2 \leq 30\} \neq \emptyset$.

The set of points that are Pareto optimal, S^0 represents the area of the set of points that lie between the points of the optimum X_1^* , X_2^* , X_3^* , X_4^* . We see that in this problem, the set of admissible points S and the set of points optimal in Pareto, S^0 , are equal to each other: $S = S^0$.

Step 2. The worst unchanging part of each criterion (anti-optimum) is determined (a superscript zero):

$$\begin{aligned} X_1^0 &= \{x_1 = 3.50, x_2 = 22.0052\}, f_1^0 = f_1(X_1^0) = 2.8663; \\ X_2^0 &= \{x_1 = 2.1952, x_2 = 12.0\}, f_2^0 = f_2(X_2^0) = 13.56; \\ X_3^0 &= \{x_1 = 2.0, x_2 = 12.0\}, f_3^0 = f_3(X_3^0) = -0.1667; \\ X_4^0 &= \{x_1 = 3.5, x_2 = 12.0\}, f_4^0 = f_4(X_4^0) = -1.0847. \end{aligned} \quad (94)$$

Step 3. A systematic analysis of the set of Pareto optimal points (i.e. analysis by each criterion) is performed.

At the optimum points $X^* = \{X_1^*, X_2^*, X_3^*, X_4^*\}$, the values of the objective functions

$F(X^*) = \|f_q(X_k^*)\|_{q=1, K}^{k=1, K}$ are determined, the vector $D = (d_1 \ d_2 \ d_3 \ d_4)^T$ deviations for each criterion on the admissible set S : $d_k = f_k^* - f_k^0, k = \overline{1, 4}, d_k = \{3.017, 65.4, 0.709, 1.418\}$ $\lambda(X^*) = \|\lambda_q(X_k^*)\|_{q=1, K}^{k=1, K}$, where $\lambda_k(X) = (f_k^* - f_k^0)/d_k$.



$$F(X^*) = \begin{bmatrix} -5.883 & -79.327 & -0.0619 & 0.123 \\ -3.173 & -78.964 & -0.3038 & 0.607 \\ -3.346 & -17.206 & -0.5423 & 1.084 \\ -4.450 & -13.64 & -0.1667 & -0.333 \end{bmatrix}, \lambda(X^*) = \begin{bmatrix} 1.0 & 1.0056 & 0.3224 & 0.6776 \\ 0.1017 & 1.0 & 0.6636 & 0.3364 \\ 0.1593 & 0.0557 & 1.0 & 0.0 \\ 0.5251 & 0.0013 & 0.0 & 1.0 \end{bmatrix}, \quad (95)$$

where $X^* = \{X_1^*, X_2^*, X_3^*, X_4^*\}$.

The analysis of the values of criteria and relative estimates (95) shows that at the points of optimum X^* (diagonally) the relative estimate is equal to one. The remaining criteria are significantly less than one. It is required to find a point (parameters) at which the relative estimates are closest to unity. The solution of this problem is aimed at solving the λ -problem - step 4, 5.

Step 4. The construction of the λ -problem is carried out in two stages: a maximin optimization problem with normalized criteria is initially constructed:

$$\Lambda^0 = \max_{X \in S} \min_{k \in K} \lambda_k(X), G(X) \leq 0, X \geq 0. \quad (96)$$

At the second stage, the maximin problem (96) is transformed into a standard mathematical programming problem:

$$\lambda^0 = \max \lambda, \quad (97)$$

at restrictions

$$\lambda - (11.474 - 4.8992x_1 + 0.8868x_1^2 - 0.003x_2 + 0.0048x_2^2 - 0.0595x_1x_2 - f_1^0)/d_1 \leq 0, \quad (98)$$

$$\lambda - (-0.122 - 0.373x_1 + 0.1916x_1^2 + 0.022x_2 + 0.0006x_2^2 - 0.0173x_1x_2 - f_2^0)/d_2 \leq 0, \quad (99)$$

$$\lambda - (8.8176 - 7.68x_1 + 2.1456x_1^2 + 0.1851x_2 + 0.0894x_2^2 - 0.1454x_1x_2 - f_3^0)/d_3 \leq 0, \quad (100)$$

$$\lambda - (-0.245 - 0.747x_1 + 0.383x_1^2 + 0.044x_2 + 0.0012x_2^2 - 0.0346x_1x_2 - f_4^0)/d_4 \leq 0, \quad (101)$$

$$2.01 \leq x_1 \leq 3.5, 12.01 \leq x_2 \leq 30.01, \quad (102)$$

where the vector of unknowns has dimension of $N + 1$: $X = \{x_1, \dots, x_N, \lambda\}$, $N = 2$.

Step 5. Solution of the λ -problem. For this purpose we use the function `fmincon(...)`, [17]:
[Xo, Lo] = `fmincon('Z_TS_2Krit_L', X0, Ao, bo, Aeq, beq, lbo, ubo, 'Z_TS_LConst', options)`

The result of solving the vector optimization problem (97)-(102) with equivalent criteria, with the corresponding λ -problem (97)-(102) was obtained:

The optimum point X^0 with equivalent criteria

$$X^0 = \{X^0, \lambda^0\} = \{X^0 = \{x_1 = 2.8039, x_2 = 30.0, \lambda^0 = 0.3541\}\} \quad (103)$$

which represents the design parameters of the technological process with equivalent criteria (characteristics). The optimal point is X^0 is shown in Figure 3;

The values of the criteria $f_k(X^0)$, $k = \overline{1, K}$ at the optimum point X^0 :

$$F(X^0) = \{f_1(X^0) = 3.9345, f_2(X^0) = 77.9319, f_3(X^0) = 0.0844, f_4(X^0) = 0.1687\}; \quad (104)$$

which represent the characteristics of the technological process;

Values of relative estimates $\lambda_k(X^0)$, $k = \overline{1, K}$ at the optimum point X^0 :

$$\lambda(X^0) = \{\lambda_1(X^0) = 0.3541, \lambda_2(X^0) = 0.9842, \lambda_3(X^0) = 0.3541, \lambda_4(X^0) = 0.6459\}; \quad (105)$$

Maximum lower level $\lambda^0 = 0.3541$ at the optimum point X^0 : which is measured in relative units, among all relative estimates:

$$\lambda^0 = \min \{\lambda_1(X^0), \lambda_2(X^0), \lambda_3(X^0), \lambda_4(X^0)\} = 0.3541,$$

λ^0 – also called a guaranteed result in the relative units, i.e., $\lambda_k(X^0) \forall k \in K$ and, accordingly, the characteristics of the technological process $f_k(X^0) \forall k \in K$ it is impossible to improve, without worsening at the same time other characteristics.

In accordance with Theorem 1, at point X^0 (105), criteria 1 and 3 are contradictory. This contradiction is determined by the equality: $\lambda_1(X^0), \lambda_3(X^0) = \lambda^0 = 0.3541$, and the remaining criteria by the inequality $\{\lambda_2(X^0) = 0.9842, \lambda_4(X^0) = 0.6459\} > \lambda^0$.

2.2.6. Creation of geometrical interpretation of results of the decision in a three-dimensional coordinate.



In the allowable set of points S formed by constraints (92), the optimum points $X^* = \{X_1^*, X_2^*, X_3^*, X_4^*\}$, which are shown in Fig. 3, combined into a contour, represent the set of Pareto optimal points, $S^0 \subset S$. The coordinates of these points, as well as the characteristics of the technological process in relative units $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$ are shown in **Figure 4** in the three-dimensional space x_1, x_2 and λ , where the third axis is the relative estimate λ .

2.3. Part 2. Selection of optimal process parameters under uncertainty with the priority of the criterion, the numerical values of which are set by the system designer.

As a rule, the decision-maker is the designer of the technological process (the process of intelligent decision-making).

2.3.1. Decision-making in the technological process model at the set priority of criterion

To solve vector problems of mathematical programming (88)-(92) methods are presented, based on axiomatic normalization of criteria and principle of guaranteed result, as well as axiomatic priority of criterion, resulting from axiom 2, 3 and principle of optimality 2, which are presented in section 5, [19]. The decision maker is usually the process designer.

Step 1. We solve a vector problem (88)-(92) with equivalent criteria. The algorithm of the decision is presented in the stage 3. Numerical results of the solution of the vector problem are given above. Pareto's great number of $S^0 \subset S$ lies between optimum points $X_1^*, X_2^*, X_3^*, X_4^*, X_1^*$. This information is the basis for further research on the structure of the Pareto set. The decision maker is usually the technological process designer. If results of the solution of a vector problem with equivalent criteria do not satisfy the person making the decision, then the choice of an optimal solution is carried out from any subset of points of $S_1^0, S_2^0, S_3^0, S_4^0$, **Figure 4**.

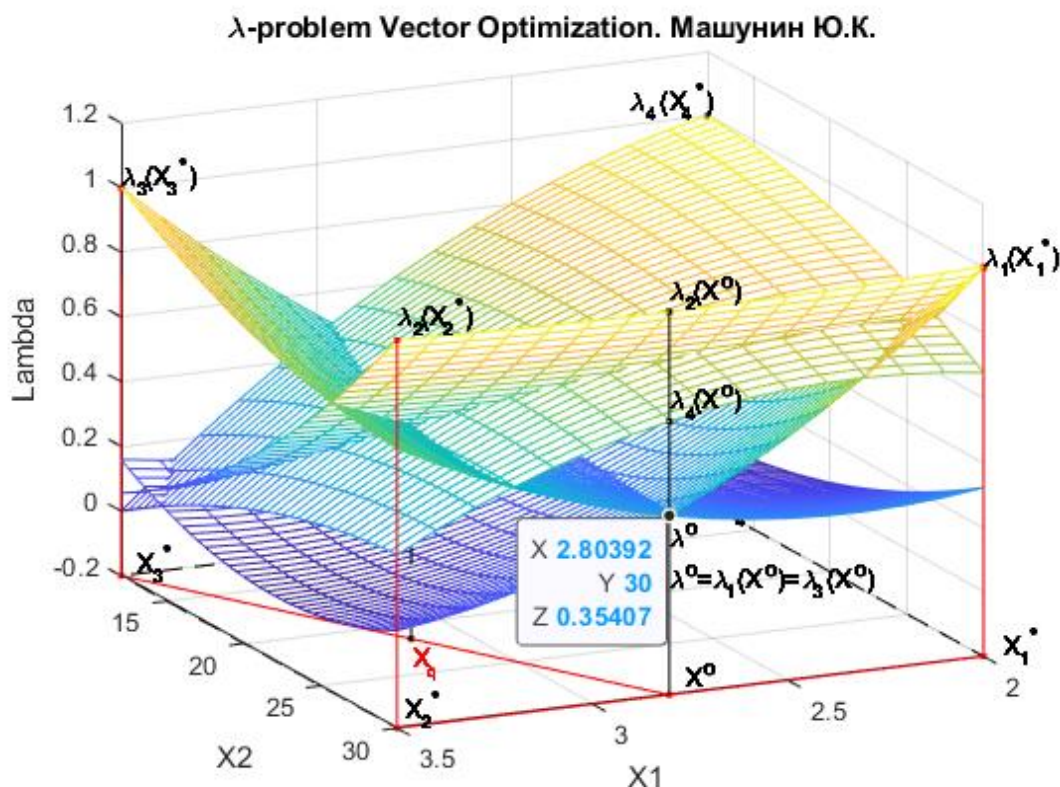


Figure 4. Results of solution λ -problems $\lambda_1(Y), \lambda_2(Y), \lambda_3(Y), \lambda_4(Y)$ in the three-dimensional coordinate system x_1, x_2 and λ



Step 2. Choice of priority criterion of $q \in K$. From the theory (the Theorem 2) it is known that in an optimum point of X^0 there are always two most contradictory criteria: $q \in K$ and $v \in K$ for which in the relative unit's precise equality is carried out:

$$\lambda^0 = \lambda_q(X^0) = \lambda_p(X^0), q, p \in K, X \in S,$$

and for the others it is carried out inequalities: $\lambda^0 \leq \lambda_k(X^0) \quad \forall k \in K$.

For the choice of the priority criterion on the display the message about results of the solution of λ -problem in physical and relative units is given: Criteria (93) in X^0 optimum point:

$$F(X^0) = \{f_1(X^0) = 3.9345, f_2(X^0) = 77.9319, f_3(X^0) = 0.0844, f_4(X^0) = 0.1687\};$$

The relative estimates in X^0 :

$$\lambda(X^0) = \{\lambda_1(X^0) = 0.3541, \lambda_2(X^0) = 0.9842, \lambda_3(X^0) = 0.3541, \lambda_4(X^0) = 0.6459\}.$$

From the function $\lambda(X^0)$ it is clear that the first and the third are the most contradictory criteria:

$$\lambda^0 = \{\lambda_1(X^0), \lambda_3(X^0)\} = 0.3541. \quad (106)$$

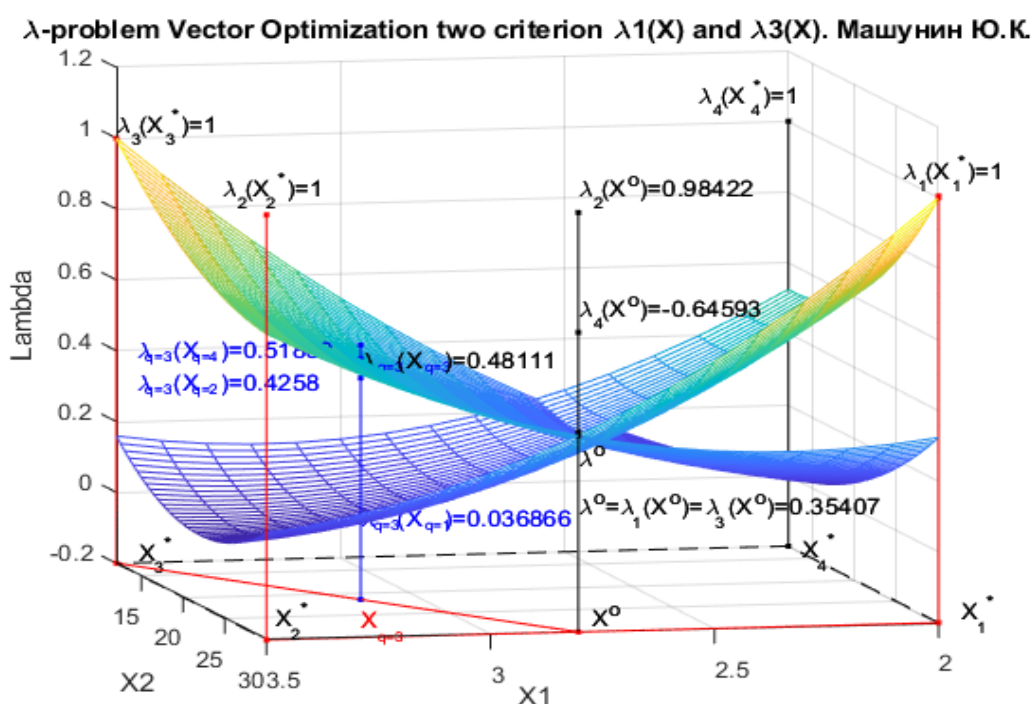


Figure 5. Result the solution of the problem (78)-(82) in the three-dimensional coordinate system $\{x_1, x_2, \lambda\}$. $\lambda_1(X^0) = \lambda_2(X^0) = \lambda_3(X^0) = 0.3542$

Select from **Figure 4** the first and third criteria and present $\{\lambda_1(X^0), \lambda_3(X^0)\}$ in the relative units in **Figure 5**.

As a rule, from a pair of the contradictory criteria, a criterion chosen by the decision maker would be improved. Such criterion is called "priority criterion", we will designate it $q = 3 \in K$. This criterion is investigated in interaction with the first criterion of $k = 1 \in K$.

On the display the message is given:

q=input ('Enter priority criterion (number) of q =') - Entered: q=3.

Step 3. Numerical limits of the change of the size of a priority of the criterion of $q = 3 \in K$ are defined.

For priority criterion of $q = 3 \in K$ changes of the numerical limits in the physical units upon transition from X^0 optimum point to the point of X_q^* received on the first step at equivalent criteria are defined. $q=3$ given about criterion are given for the screen:



$$f_q(X^0) = -0.084 \leq f_q(X) \leq -0.54 = f_q(X_q^*), q = 3 \in K. \quad (107)$$

In the relative units the criterion of $q=3$ changes in the following limits:

$$\lambda_q(X^0) = 0.35407 \leq \lambda_q(X) \leq 1 = \lambda_q(X_q^*), q = 3 \in K. \quad (108)$$

This data is analyzed.

Step 4. Choice of size of the priority criterion of $q \in K$. (Decision - making). On the message: "Enter the size of priority criterion f_q " - we enter, the size of the characteristic $q = 3 \in K$: $f_q = 0.3$.

Step 5. The relative assessment is calculated.

For the chosen size of priority criterion $f_q=0.3$ the relative assessment is calculated:

$$\lambda_q = \frac{f_q - f_q^0}{f_q^* - f_q^0} = \frac{0.30 - (-0.16670)}{0.5435 - (-0.16670)} = 0.65821. \quad (109)$$

which upon transition from X^0 point to X^* lies in the limits:

$$0.35407 = \lambda_3(X^0) \leq \lambda_3 = 0.6582 \leq \lambda_3(X_3^*) = 1, q \in K$$

Step 6. Let's calculate coefficient of the linear approximation

Assuming the linear nature of the change of the criterion of $f_q(X)$ in (107) and according to the relative assessment of λ_q , using standard linear approximation techniques, we will calculate the proportionality coefficient between $\lambda_q(X^0)$, λ_q which we will call ρ :

$$\rho = \frac{\lambda_q - \lambda_q(X^0)}{\lambda_q(X_q^*) - \lambda_q(X^0)} = \frac{0.6582 - 0.35407}{1 - 0.35407} = 0.4708, q = 3 \in K.$$

Step 7. Let's calculate coordinates of a priority of the criteria with dimension of f_q

Assuming the linear nature of change of a vector of $X^q = \{x_1 \ x_2\}$, $q = 3$ we will determine point coordinates with dimension of $f_q=0.3$, the relative assessment λ_q :

$$\begin{aligned} X_q &= \{x_1^q = x_1^0 + \rho(x_q^*(1) - x_1^0), \dots, \\ x_N^q &= x_N^0 + \rho(x_q^*(N) - x_N^0)\}, \end{aligned} \quad (110)$$

where $X^0 = \{x_1^0, \dots, x_N^0\}$, $X_q^* = \{x_q^*(1), \dots, x_q^*(N)\}$.

$X^0 = \{x_1^0 = 2.8039, x_2^0 = 30.01\}$, $X_q^* = \{x_q^*(1)=3.5, x_q^*(2)=12.01\}$.

As result of the decision (110) we will receive X_q point with coordinates:

$X^q = \{x_1 = 3.1317, x_2 = 21.525\}$.

Step 8. Calculation of the main indexes of a point of X^q .

For the received X^q point, we will calculate:

all criteria in the physical units $f_k(X^q) = \{f_k(X^q), k = \overline{1, K}\}$,

$f(X^q) = \{f_1(X^q) = 2.9775, f_2(X^q) = 41.4096, f_3(X^q) = 0.1744, f_4(X^q) = 0.3489\}$;

all relative estimates with the criterion priority

$$\lambda^q = \{\lambda_k^q, k = \overline{1, K}\}, \lambda_k(X^q) = \frac{f_k(X^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K},$$

$\lambda_k(X^q) = \{\lambda_1(X^q) = 0.0369, \lambda_2(X^q) = 0.4258, \lambda_3(X^q) = 0.4811, \lambda_4(X^q) = 0.5189\}$;

minimum relative assessment: $\min(\lambda_k(X^q)) = 0.0369$;

vector of priorities of the third criterion over other criteria:

$$P^q(X) = \{p_k^q = \frac{\lambda_q(X^q)}{\lambda_k(X^q)}, k = \overline{1, K}\},$$

$P^q = [p_1^3 = 6.2789, p_2^3 = 17.94, p_3^3 = 1.0, p_4^3 = 0.911]$;

Any point from Pareto's set $X_t^0 = \{x_t^0, X_t^0\} \in S^0$ can be similarly calculated.

Analysis of the results of the solution of the process model represented by the vector problem from the current of symmetry

In the considered vector problem of nonlinear programming with four heterogeneous criteria - the model of the technological system and the corresponding numerical variant (78)-(82) and built



on its basis λ -problems (97)-(7.32) we obtained the optimum point X^o and the maximum relative estimate λ^o :

$$X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 2.8039, x_2 = 30.0, \lambda^o = 0.3541\}\};$$

At the optimum point X^o the criteria in natural units are:

$$F(X^o) = \{f_1(X^o) = 3.9345, f_2(X^o) = 77.9319, f_3(X^o) = 0.0844, f_4(X^o) = 0.1687\},$$

and relative units:

$$\lambda(X^o) = \{\lambda_1(X^o) = 0.3541, \lambda_2(X^o) = 0.9842, \lambda_3(X^o) = 0.3541, \lambda_4(X^o) = 0.6459\};$$

This result confirms the proofs of theorem 1 about the most contradictory criteria in the vector optimization problem, i.e.

$$\lambda^o = \lambda_1(X^o) = \lambda_3(X^o) = 0.3541, \{1, 3\} \in K, X^o \in S.$$

The rest of the criteria in relative units lie within:

$$\{\lambda_2(X^o) = 0.9842, \lambda_4(X^o) = 0.6459\} > \lambda^o.$$

Theoretically, the point X^o is the center of symmetry. Indeed, the X^o point lies between the points of the optimum X_1^* and X_3^* obtained for each criterion (step 1), in which $\lambda_1(X_1^*)=1, \lambda_3(X_3^*) = 1$.

Let's show geometrically symmetry in Figure 6.

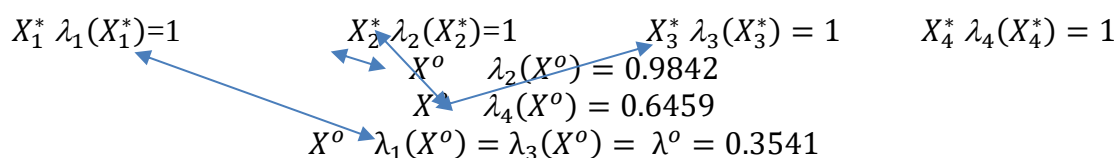


Figure 6. Geometric interpretation of symmetry in modeling of technical system with normalized criteria: $\lambda_1(X), \dots, \lambda_4(X)$.

For the two criteria of the first $\lambda_1(X)$ and the third $\lambda_3(X)$, this example considers an even numerical symmetry.

Analysis of results. The calculated size of criterion $f_q(X_t^o), q \in K$ is usually not equal to the set f_q . The error of the choice of $\Delta f_q = |f_q(X_t^o) - f_q| = |0.1744 - 0.3| = 0.125$ is defined by an error of linear approximation, $\Delta f_{q\%} = \frac{\Delta f_q}{f_q} * 100 = 40.2\%$.

In the course of the modeling parametrical restrictions can be changed, i.e. some set of optimum decisions is received. Choose a final version which in our example included from this set of optimum decisions: parameters of technological process $X^o = \{x_1 = 2.8039, x_2 = 30.0\}$;

the parameters of the technological system at a given priority criterion $q=3$:

$$X^q = \{x_1 = 3.1317, x_2 = 21.5248\}.$$

2.3.2. Geometrical interpretation of results of the decision in the technological process in three to a measured frame in physical units

We represent these parameters in a two-dimensional x_1, x_2 and three-dimensional coordinate system x_1, x_2 and λ in Figures 3, 4, 5, 6 and also in physical units for each function $f_1(X), f_2(X), f_3(X), f_4(X)$ on Figures 7, 8, 9 and 10 respectively.

The first characteristic $f_1(X)$ in physical units show in Figure 7.

Optimum points: X_1^* - maximum point; X^o - Equivalent criteria; $X_{q=3}$ - priority 3 Criteria;

X_1^o - Minimum point are the parameters of the physical characteristics of the technological process. Let us present the characteristics of the technological process in physical units:

$$f_1^* = f_1(X_1^*) = -5.8834 \leq f_1(X^o) = 3.9345 \leq f_1(X_{q=3}) = 2.9777 \leq f_1^o = f_1(X_1^o) = 2.8664.$$

The second characteristic $f_2(X)$ in physical units show in Figure 8.

Optimum points: X_2^* - maximum point; X^o - Equivalent criteria; $X_{q=3}$ - priority 3 Criteria;



X_2^0 - Minimum point are the parameters of the physical characteristics of the technological process. Let us present the characteristics of the technological process in physical units:

$$f_2^* = f_2(X_2^*) = 78.931 \leq f_2(X^0) = 77.409 \leq f_2(X_{q=3}) = 41.409 \leq f_2^0 = f_2(X_2^0) = 13.561$$

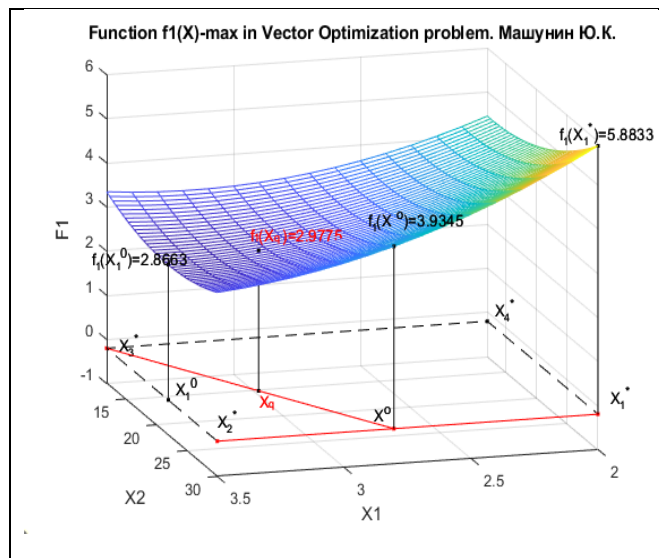


Figure 7. The first characteristics of the $f_1(X)$ technological process in physical units in the three-dimensional coordinate system $\{x_1, x_2 \text{ and } f_1(X)\}$

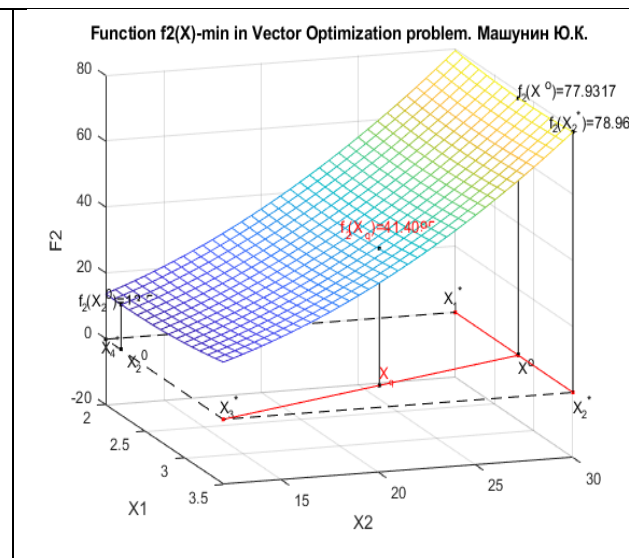


Figure 8. The second characteristics of the $f_2(X)$ technological process in physical units in the three-dimensional coordinate system $\{x_1, x_2 \text{ and } f_2(X)\}$

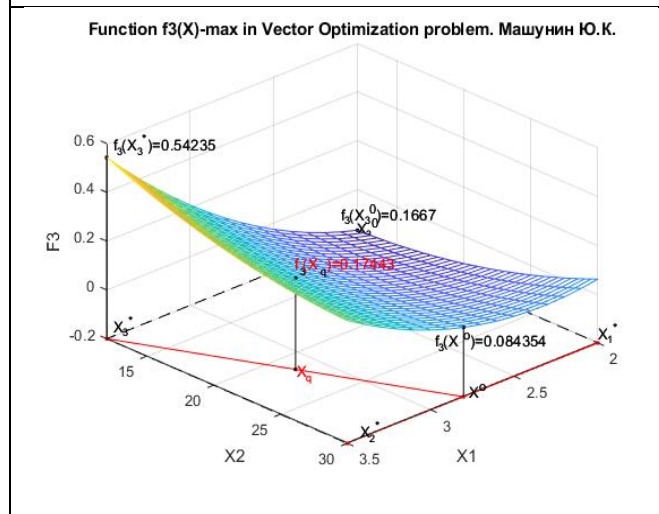


Figure 9. The third characteristics of the $f_3(X)$ technological process in physical units in the three-dimensional coordinate system $\{x_1, x_2 \text{ and } f_3(X)\}$

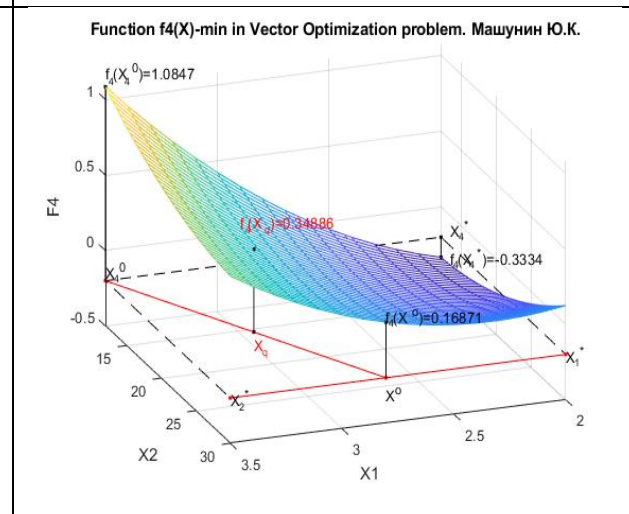


Figure 10. The fourth characteristics of the $f_4(X)$ technological process in physical units in the three-dimensional coordinate system $\{x_1, x_2 \text{ and } f_4(X)\}$

The third characteristic $f_3(X)$ in physical units show in Figure 9.

Optimum points: X_3^* - maximum point; X^0 – Equivalent criteria; $X_{q=3}$ – priority 3 Criteria; X_3^0 – Minimum point are the parameters of the physical characteristics of the technological process.



Let us present the characteristics of the technological process in physical units:

$$f_3^* = f_3(X_3^*) = 0.5424 \leq f_3(X^0) = 0.0841 \leq f_3(X_{q=3}) = 0.3174 \leq f_3^0 = f_3(X_3^0) = 0.1667.$$

The fourth characteristic $f_4(X)$ in physical units show in Figure 10

Optimum points: X_4^* - maximum point; X^0 – Equivalent criteria; $X_{q=3}$ – priority 3 Criteria; X_4^0 – Minimum point are the parameters of the physical characteristics of the technological process.

Let us present the characteristics of the technological process in physical units:

$$f_4^* = f_4(X_4^*) = 0.3335 \leq f_4(X^0) = 0.1687 \leq f_4(X_{q=3}) = 0.3468 \leq f_4^0 = f_4(X_4^0) = 1.0848.$$

Similarly, Fig. 7,..., 10 the calculation and geometric interpretation of the results of the VZMP solution is formed with the priority of the second, third and fourth criteria.

Results.

$$F(X^0) = \{f_1(X^0), f_2(X^0), f_3(X^0), f_4(X^0)\};$$

- relative estimates of $\lambda(X^0) = \{\lambda_1(X^0), \lambda_2(X^0), \lambda_3(X^0), \lambda_4(X^0)\};$
- maximin λ^0 - relative level such that $\lambda^0 \leq \lambda_k(X^0) \quad \forall k \in K$

there is an optimal solution with equivalent criteria (characteristics), and the procedure for obtaining is the adoption of the optimal solution in the process with equivalent criteria (characteristics).

The second option:

- point - $X^q = \{X^{1q}, X^{2q}\};$ characteristics $f(X^q) = \{f_1(X^q), f_2(X^q), f_3(X^q), f_4(X^q)\};$
- relative estimates $\lambda_k(X^q) = \{\lambda_1(X^q), \lambda_2(X^q), \lambda_3(X^q), \lambda_4(X^q)\};$
- maximin λ^{0q} is a relative level such that $\lambda^{0q} \leq \lambda_1(X^q), \lambda_2(X^q), \lambda_3(X^q), \lambda_4(X^q).$

3. Modeling and selection of optimal parameters of a technical system in conditions of certainty and uncertainty based on multidimensional mathematics and designer's intelligence.

3.1. Mathematical Model of a Multiparameter and Multifunctional Technical System for Making an Optimal Management Decision

3.1.1. Problems of Modelling in the Design of New Technical Systems

When researching and designing new technical systems, the problem of building a mathematical model, evaluating the results of modeling and making an optimal decision based on them arises. Similar problems arise on already created technical systems (TS), which during operation must be constantly upgraded and replaced by more advanced models. The problem of modeling the vehicle (including using multi-criteria (vector) optimization) is given great attention, firstly, in domestic science, starting with the leading scientific schools of the Academy of Sciences of the USSR to the present time [4-23, 38-45], which have made a great contribution to the application of multi-criteria optimization methods, and, secondly, in foreign scientific activity, both theoretical [1, 2, 3, 24-33] and applied aspects [34-38].

When constructing a mathematical model TS, there may be situations when the functional dependence of each characteristic and constraint on the parameters of the TS is known, such a model is usually called a model in conditions of complete certainty (well-structured tasks) - while using the methods presented in the fourth chapter. Situations where there is insufficient information about the functional dependence of each characteristic and constraint on the parameters are defined as modeling under conditions of complete or partial uncertainty (poorly structured tasks) [12 - 18]. In this case, the elimination of uncertainty can go in two directions: the first is related to the use of subjective assessments and the preference of the decision-maker in assessing the options for possible solutions [22]; the second direction, determined by the qualitative and quantitative descriptions of the TS, is characterized by the use of mathematical methods for transforming information - initial data that can be obtained, for example, on the principle of "input-output". Within the framework of the second direction, this study was carried out. Currently, there are a large number of methods for evaluating experimental data and making decisions based on [34], but all of them have certain drawbacks and,



above all, those related to the incommensurability of the criteria, the analysis is presented in the works [6, 10, 22], as a result, the decision is made on an intuitive basis. Therefore, it is important to develop new methods for modeling the vehicle, evaluating the initial data and making decisions based on them.

3.1.2. Mathematical model of technical systems in the conditions of certainty

The technical system which functioning depends on N - a set of design data is considered:

$X = \{x_1 x_2 \dots x_N\}$, or $X = \{x_j, j = \overline{1, N}\}$, N - number of parameters, each of which lies in the set limits: $x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1, N}$, or $X^{min} \leq X \leq X^{max}$, where $x_j^{min}, x_j^{max}, \forall j \in N$ - lower and top limits of change of the vector of parameters of the technical system.

The result of functioning of the technical system is defined by a set K to technical characteristics of $f_k(X), k = \overline{1, K}$, which functionally depend on design data $X = \{x_j, j = \overline{1, N}\}$, in total they represent a vector function:

$F(X) = (f_1(X) f_2(X) \dots f_K(X))^T$, on which functional limitations are imposed:

$$f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1, K} \text{ or } G(X) \leq B.$$

The set of characteristics (criteria) to is subdivided into two subsets K_1 and K_2 : $K = K_1 \cup K_2$ is a subset of technical characteristics, the numerical values of which are desirable to obtain as high as possible: $f_k(X) \rightarrow max, k = \overline{1, K_1}$.

K_2 - it subsets of technical characteristics which numerical sizes it is desirable to receive as it is possible below: $f_k(X) \rightarrow min, k = \overline{1, K_2}, K_2 \equiv \overline{K_1 + 1, K}$.

The mathematical model should, firstly, the purposes of the technical system which are presented by the characteristics of $F(X) = \{f_k(X), k = \overline{1, K}\}^T$ secondly, to consider $X^{min} \leq X \leq X^{max}$ restrictions. The mathematical model of the technical system which solves in general a problem of a choice of the optimum design decision (a choice of optimum parameters), we will present in the form of a vector problem of mathematical programming.

$$Opt F(X) = \{max F_1(X) = \{max f_k(X), k = \overline{1, K_1}\}, \quad (111)$$

$$min F_2(X) = \{min f_k(X), k = \overline{1, K_2}\}\}, \quad (112)$$

$$\text{the restrictions: } G(X) \leq 0, f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1, K}, \quad (113)$$

$$x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1, N}, \quad (114)$$

where X is the vector of controlled variables (constructive parameters);

$F(X) = \{f_k, k = \overline{1, K}\}$ - vector criterion which everyone a component submits the characteristic of the TS (1a)-(2a) which is functionally depending on the vector of variables X ; in (3a) $G(X) = \{g_1(X) g_2(X) \dots g_M(X)\}^T$ - vector-a function of the restrictions imposed on functioning of the technical system, M - a set of restrictions.

Restrictions are defined proceeding in them technological, physical and to that similar processes and can be presented by functional restrictions, for example:

$$f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1, K}.$$

It is supposed that the $f_k, k = \overline{1, K}$ functions are differentiated and convex, $g_i(X), i = \overline{1, M}$, are continuous, and (113)-(114) set of admissible points of S set by restrictions isn't empty and represents a compact: $S = \{X \in R^n | G(X) \leq 0, X^{min} \leq X \leq X^{max}\} \neq \emptyset$.

Criteria and restrictions (111)-(114) form mathematical model of technical system. It is required to find such vector of the $X^0 \in S$ parameters at which everyone a component the vector - functions $F(X): F_1(X) = \{f_k(X), k = \overline{1, K_1}\}$ accepts the greatest possible value, and a vector - functions $F_2(X) = \{f_k(X), k = \overline{1, K_2}\}$ are accepted by the minimum value.

3.1.3. Problems of Constructing a Model of a Technical System in the Conditions of Certainty and Uncertainty in the Aggregate

When constructing a mathematical model of TS (111)-(114), two options are possible. The first option, when the functional dependence of each criterion (1a)-(2a), restrictions (113)-(114) on



the parameters imposed on functioning is known, it is customary to call it a model under conditions of certainty, which are represented by model (111)-(114). The second option is when there is insufficient information about the functional dependence of each criterion (111)-(112) and restrictions (113)-(114) on the parameters. Only discrete (usually experimental) data are known:

$$X_i = \begin{bmatrix} x_{i1}, \dots, x_{iN} \\ \dots \\ x_{i1}, \dots, x_{iN} \end{bmatrix}, \quad i = \overline{1, M}, \text{ where } N \text{ is a set of parameters, } M \text{ is a set of data, or}$$

$X_i = \{x_{ij}, j = \overline{1, N}, i = \overline{1, M}\}$. A corresponding discrete set of characteristics is also presented:

$$I_i(X) = \{(y_k(X_i), i = \overline{1, M})\}^T = \begin{bmatrix} y_{i1}(X_i), \dots, y_{iK}(X_i) \\ \dots \\ y_{i1}(X_i), \dots, y_{iK}(X_i) \end{bmatrix}.$$

Such a state is represented by the modelling of a technical system under conditions of uncertainty. To obtain data $I_i(X) = \{(y_k(X_i), i = \overline{1, M})\}^T$ experimental studies of the technical system are carried out on the principle of "input-output", while the problem of decision-making under conditions of uncertainty is formed and the problem of choosing the optimal estimate based on the data obtained arises.

3.1.4. Constructing a Mathematical Model of a Technical System in Conditions of Certainty and Uncertainty in the Aggregate

In real life, the conditions of certainty and uncertainty are combined. The model of the technical system should also reflect these conditions. Using the symbols of the mathematical model, we transform the model (111) - (114) taking into account the uncertainty conditions, as a result we get:

"Model of a technical system in conditions of certainty and uncertainty":

$$Opt F(X) = \{max F_1(X) = \{max f_k(X), k = \overline{1, K_1^{def}}\}, \quad (115)$$

$$max I_1(X) \equiv \{max x_k(X_i), \quad i = \overline{1, M}\}^T, k = \overline{1, K_1^{unc}}, \quad (116)$$

$$min F_2(X) = \{min f_k(X), k = \overline{1, K_2^{def}}, \quad (117)$$

$$min I_2(X) \equiv \{min x_k(X_i), \quad i = \overline{1, M}\}^T, k = \overline{1, K_1^{unc}}, \quad (118)$$

$$\text{the restrictions } f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1, K}, \quad (119)$$

$$x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1, N}, \quad (120)$$

where $X = \{x_j, j = \overline{1, N}\}$ - vector of controlled variables (parameters) TS;

$F(X) = \{F_1(X) F_2(X) I_1(X) I_2(X)\}$ - a vector criterion, each component of which represents a vector of criteria (characteristics) of a technical system (5a) - (8a), which functionally depend on the discrete values of the variable vector, where in X (115) and (117) K_1^{def} , K_2^{def} (definiteness), а в (116) и (118) K_1^{unc} , K_2^{unc} (uncertainty) a set of criteria max and min formed in conditions of certainty and certainty.

3.1.5. Example. Simulation and Optimization of Parameters of Magnetoelectric Linear Inductor Electric Motors (LD) on DC

The content class of technical systems that can be represented by a vector problem (115) - (120) can be attributed to a fairly large number of their tasks from various sectors of the state economy: electrical, aerospace, metallurgical (selection of the optimal structure of the material), chemical, etc.

As an example, let's present the work of V. L. Levitsky "Modeling and optimization of parameters of magnetoelectric linear inductor electric motors (LD) of direct current" [7, pp.50-120]. A construct was designed - forced LD (FLD), the model of which was reduced to a vector problem of mathematical programming (12) - (114). In this case, the vector of design parameters $X = \{x_1, x_2, \dots, x_5\}$ consisted of: x_1 - air gap δ , x_2 - toothed pitch, x_3 - numbers of teeth, x_4 - hub height, x_5 - pole overlap coefficient. Vector of design criteria $F(X) = f(X), p(X), \vartheta(X), \dots$ included: $f(X)$ - rated tractive effort, $p(X)$ - rated power, $\vartheta(X)$ - nominal efficiency, etc. p., a total of ten indicators. To construct the dependencies $F(X)$ on these design parameters, the central orthogonal plan of the



second order was used [7 p.96]. From another branch, we will present the work [23] - "... multi-criteria optimization of static modes of mass transfer processes on the example of absorption in the production of gas separation". Thus, experimental data, both from the FLD problem and similar vehicles from other industries, can be presented in the form of a theoretical (systemic) problem (111) - (114).

In this article, the technical system is considered in statics. But technical systems can be considered in dynamics, using differential-difference methods of transformation, conducting research for a small discrete period $\Delta t \in T$.

3.2. Part 1. The choice of optimal parameters of engineering systems in conditions of certainty are equivalent criteria.

The first stage, as well as the stage of analyzing the results of the solution, choosing the priority criterion and its value, is performed by **the constructor of the technical system** of a complex structure. The remaining stages are performed by **a mathematician-programmer**.

3.2.1. The technical assignment: Construction of a mathematical and numerical model of the technical system of a complex structure.

It is given. The functioning of a technical system (TS) is determined by four parameters

$X = \{x_1, x_2, x_3, x_4\}$, which represent a vector of controlled variables. The parameters of the vehicle are set within the following limits:

$$21 \leq x_1 \leq 79, 5 \leq x_2 \leq 59, 2.1 \leq x_3 \leq 9, 2.2 \leq x_4 \leq 7. \quad (121)$$

The functioning of the vehicle is determined by four characteristics (criteria)

$F(X) = \{f_1(X), f_2(X), f_3(X), f_4(X)\}$, the value of which depends on the vector of the parameters $X = \{x_j, j = \overline{1, N}, N = 4\}$.

Conditions of certainty. They are characterized by the fact that for the first characteristic $h_1(Y)$ the functional dependence on the parameters $X = \{x_j, j = \overline{1, N}, N = 4\}$ is known.

$$f_1(X) = 323.84 - 2.249x_1 - 3.49x_2 + 10.7267x_3 + 13.124x_4 + 0.0968x_1x_2 - 0.062x_1x_3 - 0.169x_1x_4 + 0.0743x_2x_3 - 0.1042x_2x_4 - 0.0036x_3x_4 + 0.0143x_1^2 + 0.0118x_2^2 - 0.2434x_3^2 - 0.5026x_4^2. \quad (122)$$

Uncertainty conditions. They are characterized by the fact that for the second, third, and fourth characteristics: $f_2(X), f_3(X), f_4(X)$ the results of experimental data are known: the values of the parameters and the corresponding characteristics.

Numerical values of parameters X and characteristics of $f_2(X), f_3(X), f_4(X)$ are presented in table 1.

Table 1.

Numerical values of components and characteristics of the technical system.

x_1	x_2	x_3	x_4	$f_2(X)$	$f_3(X)$	$f_4(X)$
20	0	2	2	1149.6	115.1	24.24 27.60
20	0	2	5	1164.0	114.5	28.80 30.00
20	0	2	8	1176.0	114.4	31.20 32.40
20	0	5	2	1212.0	118.8	33.60 34.80
20	0	5	5	1260.0	113.8	34.80 19.92
20	0	5	8	1257.6	113.3	21.60 25.20
20	0	8	2	1256.4	110.7	29.76 33.48
20	0	8	5	1252.8	109.2	37.20 39.48
20	0	8	8	1251.6	108.5	42.00 49.20
20	30	2	2	2143.2	128.3	15.60 18.00
20	30	2	5	2154.0	127.4	21.60 24.24
20	30	2	8	2163.6	126.8	28.80 32.40
20	30	5	2	2176.8	126.1	35.16 39.60



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Направление: Физико-математические науки

20	30	5	5	2185.2	124.3	44.88	11.28
20	30	5	8	2198.4	124.1	14.40	16.80
20	30	8	2	2211.6	123.9	21.12	22.80
20	30	8	5	2232.0	121.4	27.60	30.84
20	30	8	8	2245.2	121.7	36.00	40.56
20	60	2	2	2954.4	150.4	52.80	60.00
20	60	2	5	2820.0	144.9	64.80	68.64
20	60	2	8	2772.0	140.8	75.60	82.80
20	60	5	2	2748.0	138.6	88.08	97.20
20	60	5	5	2832.0	140.8	107.64	
20	60	5	8	2904.0	143.5	40.56	45.60
20	60	8	2	3022.8	146.0	52.80	60.00
20	60	8	5	3036.0	144.9	67.20	73.20
20	60	8	8	3056.4	143.8	79.44	85.20
50	0	2	2	3583.2	181.3	99.00	31.92
50	0	2	5	3601.2	180.8	36.00	43.20
50	0	2	8	3608.4	179.4	51.36	61.20
50	0	5	2	3616.8	179.1	72.00	82.80
50	0	5	5	3622.8	178.0	86.40	90.36
50	0	5	8	3637.2	177.6	23.28	30.00
50	0	8	2	3651.6	176.9	36.00	42.72
50	0	8	5	3672.0	175.3	48.00	54.00
50	0	8	8	36852	174.7	62.16	73.20
50	30	2	2	1195.2	123.6	81.72	87.00
50	30	2	5	1212.0	118.7	94.80	
50	30	2	8	1236.0	115.9	103.20	
50	30	5	2	1251.6	115.1	116.16	
50	30	5	5	1272.0	113.2	126.00	
50	30	5	8	1296.0	111.8	136.80	
50	30	8	2	1318.8	110.7	145.44	
50	30	8	5	1344.0	108.2	156.00	
50	30	8	8	1388.4	106.3	174.72	
50	60	2	2	2176.8	132.8		
50	60	2	5	2196.0	131.1		
50	60	2	8	2220.0	129.7		
50	60	5	2	2245.2	128.3		
50	60	5	5	2286.0	127.0		
50	60	5	8	2294.4	125.6		
50	60	8	2	2313.6	123.9		
50	60	8	5	2340.0	114.5		
50	60	8	8	2382.0	119.5		
80	0	2	2	2988.0	154.8		
80	0	2	5	3012.0	153.2		
80	0	2	8	3036.0	151.8		
80	0	5	2	3056.4	150.4		
80	0	5	5	3108.0	150.7		
80	0	5	8	3156.0	151.2		
80	0	8	2	3244.8	151.5		



80	0	8	5	3228.0	144.9	
80	0	8	8	3193.2	140.8	
80	30	2	2	3616.8	185.7	
80	30	2	5	3639.6	183.5	
80	30	2	8	3660.0	182.2	
80	30	5	2	3685.2	181.3	
80	30	5	5	3708.0	179.4	
80	30	5	8	3732.0	178.0	
80	30	8	2	3753.6	176.9	
80	30	8	5	3672.0	175.3	
80	30	8	8	3822.0	172.5	
80	60	2	2	1218.0	128.3	
80	60	2	5	1248.0	125.6	
80	60	2	8	1272.0	124.2	
80	60	5	2	1318.8	121.7	
80	60	5	5	1344.0	118.7	
80	60	5	8	1392.0	115.9	
80	60	8	2	1422.0	115.1	
80	60	8	5	1464.0	110.4	
80	60	8	8	1524.0	108.5	
$\min f_i(X), i=1, \dots, 81$				1149.6	92.4	11.3
$\max f_i(X), i=1, \dots, 81$				3822.0	161.5	174.7

On the parameters $X = \{x_1, x_2, x_3, x_4\}$ are in total imposed restriction:

$$x_1 + x_2 + x_3 + x_4 = 102.$$

In the made decision, assessment size on the first and third characteristic $f_1(X), f_3(X)$ it is desirable (criterion), to receive as above $f_1(X) \rightarrow \max f_3(X) \rightarrow \max$, on second and fourth characteristic $f_2(X), f_4(X)$ as low as possible is possible $f_2(X) \rightarrow \min f_4(X) \rightarrow \min$.

In the pilot studies the rate of an increase of parameters:

$X = \{x_1, x_2, x_3, x_4\}$ will be defined in the following limits:

$$x_1 \in [20. 50. 80.]; x_2 \in [0. 30. 60.]; x_3 \in [2. 5. 8.], x_4 \in [2. 5. 8.], \quad (123)$$

It is required. 1) To construct mathematical model of the technical system in the form of a VPMP.

2) To carry out model operation: first, on the basis of the constructed mathematical model of a vector problem, secondly, on the basis of methods of solution of a VPMP of non-linear programming at equivalent criteria, and, thirdly, the software developed for these purposes in the MATLAB system. To make an optimal solution: The choice of optimum composition of the technical system according to its functional characteristics taking into account their equivalence. 4) Choose a priority criterion. Solve the vector optimization problem and make the best (optimal) solution with the given priority of the criterion. 5) To present a geometric interpretation of the results of the solution in the design of an engineering system for the transition from two-dimensional to N-dimensional space in relative units. 6) To show the geometric interpretation of the solution results in the design of an engineering system for the transition from two-dimensional to N-dimensional space in physical units.

Note: The author has developed software in the MATLAB system for solving vector problems of mathematical programming. The vector problem includes four variables (parameters of the technical system): $X = \{x_1, x_2, x_3, x_4\}$ and four criteria (characteristics)

$F(X) = \{f_1(X), f_2(X), f_3(X), f_4(X)\}$. But for each new data (new system), the program is configured individually. In software, the criteria are $F(X) = \{f_1(X), \dots, f_4(X)\}$ uncertainty conditions (Table 1).



Stage 1a. Mathematical model of the technical system under conditions of certainty

The construction of a mathematical model for making an optimal managerial decision on the structure of the technical system is shown in Section 1. In accordance with (115)-(120), let us present a mathematical model of the technical system 1 under conditions of certainty and uncertainty in the aggregate in the form of a vector problem:

$$Opt F(X) = \{ \max F_1(X) = \{ \max f_k(X), k = \overline{1, K_1^{def}} \}, \quad (124)$$

$$\max I_1(X) \equiv \{ \max f_k(X_i, i = \overline{1, M})^T, k = \overline{1, K_1^{unc}} \}, \quad (125)$$

$$\min F_2(X) = \{ \min f_k(X), k = \overline{1, K_2^{def}} \}, \quad (126)$$

$$\min I_2(X) \equiv \{ \min f_k(X_i, i = \overline{1, M})^T, k = \overline{1, K_2^{unc}} \}, \quad (127)$$

$$\text{at restrictions: } f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1, K}, \quad (128)$$

$$x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1, N},$$

where $X = \{x_j, j = \overline{1, N}\}$ - vector of controlled variables (structural parameters of the technical system);

$F(X) = \{F_1(X) F_2(X) I_1(X) I_2(X)\}$ represents a vector criterion, each component of which is a vector of criteria (characteristics) of the technical system, which are functionally dependent on the discrete values of the vector of variables;

$F_1(X) = \{f_k(X), k = \overline{1, K_1^{def}}\}$, $F_2(X) = \{f_k(X), k = \overline{1, K_2^{def}}\}$ is a set of *max* and *min* functions, respectively;

$$I_1(Y) = \{ \{f_k(X_i, i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}} \},$$

$I_2(Y) = \{ \{f_k(X_i, i = \overline{1, M})\}^T, k = \overline{1, K_2^{unc}} \}$ is the set of matrices *max* and *min*, respectively; (definiteness) K_1^{unc}, K_2^{unc} (uncertainty) a set of criteria *max* and *min* formed under conditions of certainty and uncertainty;

in (128) $f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1, K}$ is represented by the vector function of the constraints imposed on the functioning of the technical system;

in (128) $x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1, N}$ – parametric constraints.

Let's assume that the functions $f_k(X), k = \overline{1, K}$ are differentiable and convex, $g_i(X), i = \overline{1, M}$ are continuous, and the set of valid points S given by constraints (128) is not empty:

$$S = \{X \in R^n | G(Y) \leq 0, X^{min} \leq X \leq X^{max}\} \neq \emptyset, \text{ and is a compact.}$$

3.2.2. Stage 2. Numerical Model of the technical system under Conditions of Certainty

Certainty conditions are characterized by the functional dependence of each characteristic and constraint of the technical system. In the example, the characteristic (124) and constraints (128) are known. Using the information data (121), (122) we construct a single-criteria problem of nonlinear programming under conditions of certainty:

$$\max f_1(X) = 323.84 - 2.249x_1 - 3.49x_2 + 10.7267x_3 + 13.124x_4 + 0.0968x_1x_2 - \\ 0.062x_1x_3 - 0.169x_1x_4 + 0.0743x_2x_3 - 0.1042x_2x_4 - 0.0036x_3x_4 + 0.0143x_1^2 + \\ 0.0118x_2^2 - 0.2434x_3^2 - 0.5026x_4^2, \quad (129)$$

$$\text{at restrictions: } x_1 + x_2 + x_3 + x_4 = 102, \\ 21 \leq x_1 \leq 79, 5 \leq x_2 \leq 59, 2.1 \leq x_3 \leq 9.0, 2.2 \leq x_4 \leq 7.0. \quad (130)$$

This data is further used to build a mathematical model of the technical system.

3.2.3. Stage 3. Transformation of experimental data (uncertainty conditions) into data with functional dependence (certainty conditions)

Conditions of uncertainty are characterized by the fact that the initial data characterizing the object under study are represented by: a) random, b) fuzzy, or c) incomplete data, i.e., under conditions of uncertainty, only a finite set of measured parameters $X = \{x_j, j = \overline{1, N}\}$ are known: $X_v = \{x_{iv}, v = \overline{1, V}, i = \overline{1, M}\}$, where $v = \overline{1, V}$ - the number of parameters that are studied in the



model of the technical system, $i = \overline{1, M}$ - number and set of data; A set of K characteristics of a technical system: $f_k(X_v) = \{x_{iv}, v = \overline{1, V}, i = \overline{1, M}, k = \overline{1, K}\}$.

Therefore, under conditions of uncertainty, there is not enough information about the functional dependence of each characteristic and constraints on the parameters. The information data of options a) and b) shall be converted into numerical data of option c) and shall be presented in tabular form. The paper considers option c) information with incomplete data, which, as a rule, is obtained as **a result of an experiment**.

Taking into account the measured parameters, X_v and the corresponding set of K characteristics: $f_k(X_v) = \{x_{iv}, v = \overline{1, V}, i = \overline{1, M}, k = \overline{1, K}\}$. Let us present a matrix of the results of experimental data on the technical system under study:

$$\begin{aligned} &\text{at restrictions: } x_1 + x_2 + x_3 + x_4 = 102, \\ &21 \leq x_1 \leq 79, 5 \leq x_2 \leq 59, 2.1 \leq x_3 \leq 9.0, 2.2 \leq x_4 \leq 7.0. \end{aligned} \quad (130)$$

$$I = \begin{bmatrix} a_1 \\ \dots \\ a_M \end{bmatrix} = \begin{bmatrix} X_1 = x_{11}, x_{12}, x_{13}, x_{14} & f_2(X_1), f_3(X_1), f_4(X_1) \\ \dots & \dots \\ X_M = x_{M1}, x_{M2}, x_{M3}, x_{M4} & f_2(X_M), f_3(X_M), f_4(X_M) \end{bmatrix}, \quad (131)$$

Let us present a mathematical model of the technical system under uncertainty in the form of a vector problem of mathematical programming:

$$\text{Opt } F(X) = \{\max I_1(Y) \equiv \{\max f_k(X_i, i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}, \quad (132)$$

$$\min I_2(Y) \equiv \{\min f_k(X_i, i = \overline{1, M})\}^T, k = \overline{1, K_2^{unc}}\}, \quad (133)$$

$$\text{at restriction } f_k^{\min} \leq f_k(X) \leq f_k^{\max}, k = \overline{1, K}, \quad (134)$$

$$\sum_{v=1}^V x_v(t) = a, x_v^{\min} \leq x_v \leq x_v^{\max}, v = \overline{1, V}, \quad (135)$$

where $X = \{x_v, v = \overline{1, V}\}$ is the vector of controlled variables (parameters);

$F(X) = \{I_1(X) I_2(X)\}$ is a vector criterion, each component of which represents a vector of criteria (output characteristics of the object under study). The value of the characteristic (function) depends on the discrete values of the vector of the variables X . $I_1(X) = \overline{1, K_1^{unc}}$, $I_2(X) = \overline{1, K_2^{unc}}$ (uncertainty) – a set of max and min criteria formed under uncertainty;

in (134) $f_k^{\min} \leq f_k(X) \leq f_k^{\max}, k = \overline{1, K}$ – vector-function of the restrictions imposed on the functioning of the object under study, $x_v^{\min} \leq x_v \leq x_v^{\max}, v = \overline{1, V}$ – parametric constraints of the object under study.

3.2.4. Stage 4. Construction of an Aggregated Mathematical and Numerical Model of the Structure of the Material under Uncertainty.

Formation under uncertainty consists in the use of qualitative, quantitative descriptions of the technical system, obtained according to the principle of "input-output" in Table 2.

Transformation of initial data (information): $f_2(X_i, i = \overline{1, M}), f_3(X_i, i = \overline{1, M}), f_4(X_i, i = \overline{1, M})$ in functional form: $f_2(X), f_3(X), f_4(X)$ is carried out by using mathematical methods of regression analysis.

The initial data are generated in Table 2 in the MATLAB system in the form of a matrix:

$$I = |X, F| = \{x_{i1}, x_{i2}, x_{i3}, x_{i4}, f_{i2}, f_{i3}, f_{i4}, i = \overline{1, M}\}. \quad (136)$$

For each set of experimental data in Table 2 $f_k, k = 2, 3, 4$ the regression function is constructed by the least squares method $\min \sum_{i=1}^M (x_i - \bar{x}_i)^2$ in the MATLAB system.

A polynomial A_k is formed, which determines the relationship between the parameters

$X_i = \{x_{i1}, x_{i2}, x_{i3}, x_{i4}\}$ and the $\bar{f}_{kl} = f(X_i, A_k), k = 2, 3, 4$.

The result of the formation is a system of coefficients:

$A_k = \{A_{0k}, A_{1k}, \dots, A_{14k}\}$, that define the coefficients of a quadratic polynomial:

$$f_k(X, A) = A_{0k} + A_{1k}x_1 + A_{2k}x_2 + A_{3k}x_3 + A_{4k}x_4 + A_{5k}x_1x_2 + A_{6k}x_1x_3 + A_{7k}x_1x_4 + A_{8k}x_2x_3 + A_{9k}x_2x_4 + A_{10k}x_3x_4 + A_{11k}x_1^2 + A_{12k}x_2^2 + A_{13k}x_3^2 + A_{14k}x_4^2, k = 2, 3, 4.$$



Polynomial approximation software with four variables and fourteen factors has been developed. As a result, the experimental data of Table 2 are transformed by the system of coefficients of the three functions $f_k(X, A)$ in the form of a table (Program: Z_Material_MMTT32_os13_4k):

$$\begin{aligned} A_0 &= [323.8408 \ 954.8634 \ 110.02 \ 21.0051 \\ &\quad -3.4938 \ 37.0392 \ 0.6206 \ -0.8403 \\ &\quad 10.7267 \ -31.0303 \ -0.4287 \ -0.4314 \\ &\quad 13.1239 \ -54.0031 \ -2.5176 \ 1.1718 \\ &\quad 0.0969 \ -0.9219 \ -0.0151 \ 0.0166 \\ &\quad -0.0621 \ 0.5644 \ -0.0094 \ 0.0850 \\ &\quad -0.1696 \ 0.8966 \ 0.0222 \ -0.0001 \\ &\quad 0.0743 \ -0.1540 \ -0.0198 \ 0.0522 \\ &\quad -0.1042 \ 0.3919 \ 0.0184 \ 0.0003 \\ &\quad 0.0036 \ -0.0135 \ -0.0006 \ 0.0006 \\ &\quad 0.0142 \ 0.0477 \ -0.0004 \ -0.0021 \\ &\quad 0.0117 \ 0.0437 \ -0.0003 \ 0.0035 \\ &\quad -0.2433 \ 3.8489 \ 0.0390 \ 0.0061 \\ &\quad -0.5026 \ 3.1748 \ 0.1414 \ -0.0310]; \\ R_j &= [0.6115 \ 0.7149 \ 0.6551 \ 0.9017]; \\ RR_j &= [0.3740 \ 0.5111 \ 0.4292 \ 0.8130]. \end{aligned} \quad (137)$$

On the basis of $A_0(2)$ $A_0(3)$ $A_0(4)$, the functions $f_2(X)$, $f_3(X)$ and $f_4(X)$ are constructed, which, taking into account the obtained coefficients, will take the form:

$$\max f_3(X), \min f_2(X), \min f_4(X), \text{ при ограничениях: } (130). \quad (138)$$

Minimum and maximum values of experimental data are $x_1, \dots, x_4$. are presented at the bottom of Table 2. The minimum and maximum values of the functions $f_1(X)$, $f_2(X)$, $f_3(X)$ and $f_4(X)$ differ slightly from the experimental data.

The correlation index and coefficients of determination are presented in the lower rows of Table 2. The results of regression analysis (137), (138) are further used in the construction of a mathematical model of the technical system.

The resulting mathematical model in the form of VPMP (137), (138) is a numerical model of the structure of the technical system under uncertainty (Numerical data of the model (138) are presented below).

3.2.5. Construction of an aggregated numerical model of the structure of the technical system under conditions of certainty

Combining mathematical models of the technical system under conditions of certainty (129)-(130) and uncertainty (138), we will present a mathematical model of the technical system under conditions of certainty and uncertainty in the aggregate in the form of a vector problem:

$$\begin{aligned} Opt F(X) &= \{ \max F_1(Y) = \{ \max f_k(X), k = \overline{1, K_1^{def}} \}, \\ &\quad \max I_1(Y) \equiv \{ \max f_k(X_i, i = \overline{1, M}) \}^T, k = \overline{1, K_1^{unc}} \}, \\ &\quad \min F_2(Y) = \{ \min f_k(X), k = \overline{1, K_2^{def}} \}, \\ &\quad \min I_2(X) \equiv \{ \min f_k(X_i, i = \overline{1, M}) \}^T, k = \overline{1, K_1^{unc}} \}, \end{aligned} \quad (139)$$

at restriction $f_k^{min} \leq f_k(X) \leq f_k^{max}, k = \overline{1, K}, x_j^{min} \leq x_j \leq x_j^{max}, j = \overline{1, N}$,
where $X = \{x_j, j = \overline{1, N}\}$ - vector of controlled variables (design parameters);

$F_1(X) = \{f_k(X), k = \overline{1, K_1^{def}}\}, F_2(X) = \{f_k(X), k = \overline{1, K_2^{def}}\}$ –

a set of *max* and *min* features respectively;

$I_1(Y) = \{\{f_k(X_i, i = \overline{1, M})\}^T, k = \overline{1, K_1^{unc}}\}, I_2(X) = \{\{f_k(Y_i, i = \overline{1, M})\}^T, k = \overline{1, K_2^{unc}}\}$



a set of *max* and *min* matrices, respectively; (*definiteness*), K_1^{unc} , K_2^{unc} (*uncertainty*) is a set of criteria *max* and *min* formed under conditions of certainty and uncertainty. Combining the numerical models of the technical system under conditions of certainty (136) and uncertainty (138), let us present the numerical model of the technical system under conditions of certainty and uncertainty in the aggregate in the form of a vector problem:

$$opt F(X) = \{max F_1(X) = \{max f_1(X) \equiv 323.84 - 2.25x_1 - 3.49x_2 + 10.72x_3 + 13.124x_4 + 0.0968x_1x_2 - 0.062x_1x_3 - 0.169x_1x_4 + 0.0743x_2x_3 -$$

$$0.1x_2x_4 - 0.0036x_3x_4 + 0.0143x_1^2 + 0.0118x_2^2 - 0.2434x_3^2 - 0.5026x_4^2,$$

$$max f_3(X) \equiv 110.22 + 0.7918x_1 + 1.73x_2 - 0.3713x_3 - 2.20x_4 - 0.0132x_1x_2 - 0.008x_1x_3 + 0.0193x_1x_4 - 0.0172x_2x_3 + 0.0161x_2x_4 - 0.0006x_3x_4 - 0.0004x_1^2 -$$

$$0.0002x_2^2 + 0.0335x_3^2 + 0.124x_4^2\},$$

$$min F_2(X) = \{min f_2(X) \equiv 954.86 + 28.67x_1 + 37.03x_2 - 31.03x_3 + 54x_4 - 0.922x_1x_2 - 2x_1x_3 + 0.896x_1x_4 - 0.154x_2x_3 + 0.3919x_2x_4 - 0.0134x_3x_4 +$$

$$0.0478x_1^2 + 0.0438x_2^2 + 3.8489x_3^2 + 3.1748x_4^2,$$

$$min f_4(X) \equiv 21.004 - 0.0097x_1 - 0.841x_2 - 0.4326x_3 + 1.1723x_4 + 0.166x_1x_2 + 0.085x_1x_3 - 0.0001x_1x_4 + 0.0523x_2x_3 + 0.0002x_2x_4 + 0.0006x_3x_4 -$$

$$0.0022x_1^2 + 0.0035x_2^2 + 0.006x_3^2 - 0.0311x_4^2\},$$

$$\text{at restrictions: } x_1 + x_2 + x_3 + x_4 = 102,$$

$$21 \leq x_1 \leq 79, 5 \leq x_2 \leq 59, 2.10 \leq x_3 \leq 9.0, 2.2 \leq x_4 \leq 7.0.$$

The VPMP (140)-(145) represents the model of optimal decision-making, i.e. the choice of the optimal of the technical system in conditions of certainty and uncertainty in the total.

3.2.6. Block 2. Making an optimal decision (selection of optimal parameters) of a technical system based on a vector problem of mathematical programming.

In order to form model of the technical system on the basis of the vector problem of mathematical programming with equivalent criteria (140)-(145), methods based on the axiomatic normalization of criteria and the principle of guaranteed result, resulting from axiom 1 and the principle of optimality 1, are presented. The solution methodology is presented in the form of a series of steps.

Step 1. Decides a vector problem of mathematical programming (140)-(145) by each criterion separately, at the same time the function *fmincon*(...) of the MATLAB system is used, the appeal to the function *fmincon*(...) is considered in [18, 19]. As a result of calculation VPMP for each criterion we receive optimum points: X_k^* and $f_k^* = f_k(X_k^*)$, $k = \overline{1, K}$ $K = 4$ sizes of criteria in this point:

Criterion 1: $X_1^* = \{x_1 = 47.6032, x_2 = 44.1968, x_3 = 8, x_4 = 2.2\}, f_1^* = f_1(X_1^*) = -392.9;$

Criterion 2: $X_2^* = \{x_1 = 56.6036, x_2 = 35.1964, x_3 = 8, x_4 = 2.2\}, f_2^* = f_2(X_2^*) = 1337.4;$

Criterion 3: $X_3^* = \{x_1 = 33.9000, x_2 = 59, x_3 = 2.1, x_4 = 7\}, f_3^* = f_3(X_3^*) = -210.5729;$

Criterion 4: $X_4^* = \{x_1 = 37.7000, x_2 = 59, x_3 = 2.1, x_4 = 2.2\}, f_4^* = f_4(X_4^*) = 32.68.$

The result the solution of a VPMP of non-linear programming (140)-(145) the optimum points $X_1^*, X_2^*, X_3^*, X_4^*$ in three-dimensional frames of x_1, x_3 and f_1, f_2, f_3, f_4 is presented on Fig. 11.



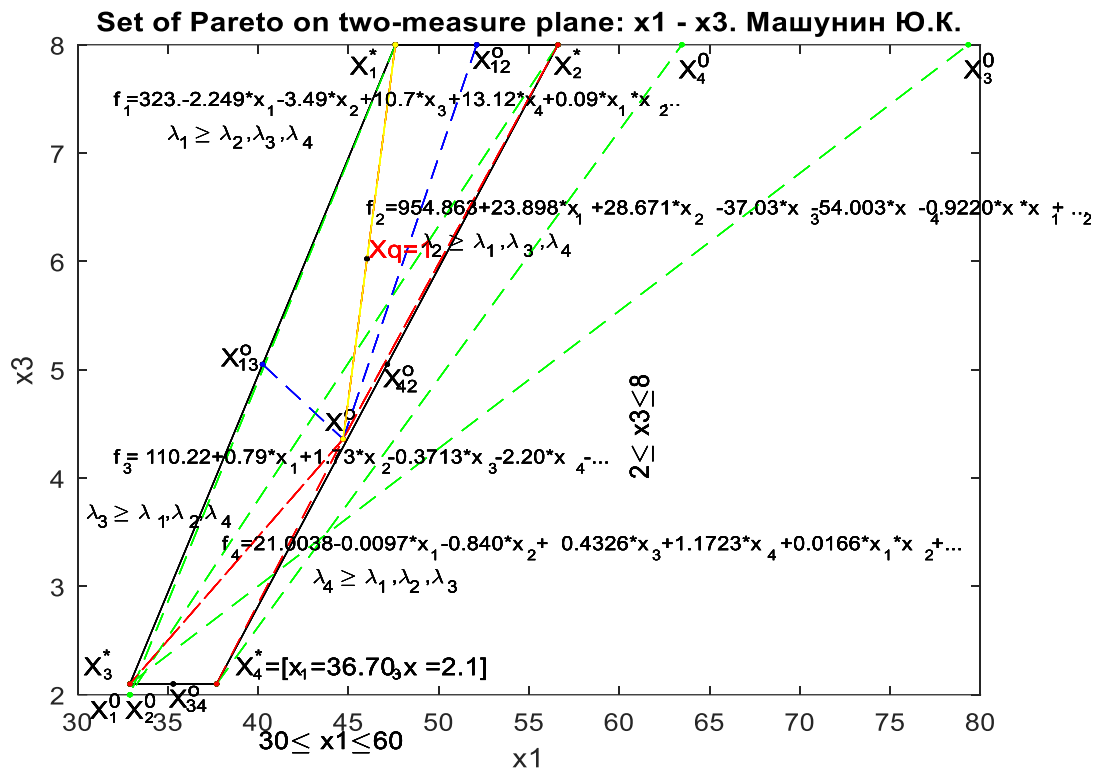


Figure 11. The permissible set of points S in the coordinates $\{x_1, x_3\}$. A set of Pareto-optimal points $S^0 \subset S$, bounded by the points X_1^* , X_2^* , X_3^* , X_4^* and the local S_1^0 between X_1^* , X_{13}^0 , X^0 , X_{12}^0 , X_1^* in the two-dimensional coordinate system $\{x_1, x_3\}$

Step 2. The worst unchangeable part of each criterion of the VPMP (140)-(145) is defined (anti-optimum) (A superscript zero):

$$\begin{aligned} X_1^0 &= \{x_1 = 32.90, x_2 = 60.0, x_3 = 2.1, x_4 = 7.00\}, f_1^0 = f_1(X_1^0) = 302.76; \\ X_2^0 &= \{x_1 = 32.90, x_2 = 60.0, x_3 = 2.1, x_4 = 7.0\}, f_2^0 = f_2(X_2^0) = -2417.52 \\ X_3^0 &= \{x_1 = 78.999, x_2 = 10.4047, x_3 = 8, x_4 = 4.5953\}, f_3^0 = f_3(X_3^0) = 170.53; \\ X_4^0 &= \{x_1 = 63.465, x_2 = 24.5125, x_3 = 8, x_4 = 6.022\}, f_4^0 = f_4(X_4^0) = -75.205. \end{aligned}$$

Step 3. Systems analysis of a set of points in the vector problem of mathematical programming that are Pareto-optimal is performed, (i.e., the analysis by each criterion). In points of an optimum of $X^* = \{X_1^*, X_2^*, X_3^*, X_4^*\}$ sizes of target functions of

$F(X^*) = \|f_q(Y_k^*)\|_{q=1, \overline{K}}^{k=1, \overline{K}}$, a vector of $D = (d_1 \ d_2 \ d_3 \ d_4)^T$ of deviations are determined by each criterion on an admissible set of S : $d_k = f_k^* - f_k^0, k = \overline{1, 4}$, and matrix of the relative estimates of $\lambda(Y^*) = \|\lambda_q(Y_k^*)\|_{q=1, \overline{K}}^{k=1, \overline{K}}$, where $\lambda_k(X) = (f_k^* - f_k^0)/d_k$

$$d_k = \{92.72, -1114.9, 41.2898, -43.1973\}.$$

$$F(X^*) = \begin{bmatrix} 392.9 & 1419 & 185.0 & 70.4 \\ 387.2 & 1337 & 178.5 & 73.8 \\ 302.8 & 2417 & 210.6 & 32.2 \\ 338.2 & 2162 & 208.0 & 32.7 \end{bmatrix}, \lambda(X^*) = \begin{bmatrix} 1.0 & 0.924 & 0.3618 & 0.1139 \\ 0.9364 & 1.0 & 0.1991 & 0.0326 \\ 0.0000 & 0.00 & 1.0 & 1.0103 \\ 0.3931 & 0.236 & 0.9364 & 1.0 \end{bmatrix}. \quad (146)$$

$$Lmin = \lambda(X^0) = \begin{bmatrix} 0 & -0.0000 & 1.0000 & 1.0103 \\ 0.0000 & 0 & 1.0000 & 1.0103 \\ -0.0276 & 0.3574 & 0 & 0.0826 \\ 0.4660 & 0.7505 & 0.0756 & 0 \end{bmatrix}$$



The analysis of sizes of criteria in the relative estimates (140)-(145) shows that at the points of the optimum $X^* = \{X_1^*, X_2^*, X_3^*, X_4^*\}$ (on diagonal) the relative assessment is equal to unit.

Other criteria (122) $\lambda(X^*) = \|\lambda_q(X_k^*)\|_{q=\overline{1,K}}^{k=\overline{1,K}}$ there is much less unit.

It is required to find such point (parameters of the material) in the VPMP at which the relative estimates are closest to unit.

The solution of this problem is directed to the solution of λ -problem - step 4. 5.

Step 4. Creation of the λ -problem is carried out in two stages. Originally, the maximine problem of optimization with the normalized criteria is under construction:

$$\lambda^0 = \max_{X \in S} \min_{k \in K} \lambda_k(X), G(X) \leq 0, X \geq 0, \quad (147)$$

which at the second stage will be transformed to a reference problem of mathematical programming (λ -problem):

$$\begin{aligned} \lambda^0 &= \max \lambda, \\ \text{at restrictions } \lambda - \frac{f_1(X) - f_1^0}{f_1^* - f_1^0} &\leq 0, \lambda - \frac{f_3(X) - f_3^0}{f_3^* - f_3^0} \leq 0, \lambda - \frac{f_2(X) - f_2^0}{f_2^* - f_2^0} \leq 0, \lambda - \frac{f_4(X) - f_4^0}{f_4^* - f_4^0} \leq 0, \end{aligned} \quad (148)$$

$$0 \leq \lambda \leq 1, 20 \leq x_1 \leq 80; 4 \leq x_2 \leq 60; 2.1 \leq x_3 \leq 9, 2.2 \leq x_4 \leq 7. x_1 + x_2 + x_3 + x_4 = 102,$$

where the vector of unknowns the λ -problem (148) has dimension of $N + 1$:

$X = \{x_1, \dots, x_N, \lambda\}, N = 4$. By substituting the numerical values of the functions $f_1(X), f_2(X), f_3(X), f_4(X)$, we get λ -problem of the following form:

$$\begin{aligned} \lambda^0 &= \max \lambda, \\ \text{at restrictions } \lambda - \frac{323.84 - 2.249x_1 - 3.49x_2 \dots - 0.2434x_3^2 - 0.5026x_4^2 - f_1^0}{f_1^* - f_1^0} &\leq 0, \\ \dots \quad \lambda - \frac{21 - 0.0097x_1 - 0.841x_2 - \dots + 0.006x_3^2 - 0.0311x_4^2 - f_4^0}{f_4^* - f_4^0} &\leq 0, \end{aligned} \quad (149)$$

$$0 \leq \lambda \leq 1, 21 \leq x_1 \leq 79; 5 \leq x_2 \leq 59; 2.1 \leq x_3 \leq 9; 2.2 \leq x_4 \leq 7. x_1 + x_2 + x_3 + x_4 = 102.$$

Step 5. Solution of the λ -problem (149).

For this purpose we use the function $fmincon(\dots)$, [16]: $[Xo, Lo] = fmincon('Z_Mater_4Krit_L', X0, Ao, bo, Aeq, beq, lbo, ubo, 'Z_Mater_LConst', options)$.

As a result of the solution of VPMP of nonlinear programming (140)-(145) at equivalent criteria and λ -problem corresponding to it (149) received:

$$X^0 = \{X^0, \lambda^0\} = \{X^0 = \{x_1 = 45.44, x_2 = 49.9, x_3 = 4.3578, x_4 = 2.2, \lambda^0 = 0.6091\} \quad (150)$$

- an optimum point – design data of the technical system, X^0 , we will present in Figure 11.

$f_k(X^0), k = \overline{1, K}$ is sizes of characteristics of the technical system (criteria):

$$\{f_1(X^0) = 369.9, f_2(X^0) = 1759.6, f_3(X^0) = 194.9, f_4(X^0) = 49.3\}; \quad (151)$$

$\lambda_k(X^0), k = \overline{1, K}$ is sizes of the relative estimates

$$\lambda(X^0) = \{\lambda_1(X^0) = 0.7452, \lambda_2(X^0) = 0.6091, \lambda_3(X^0) = 0.6091, \lambda_4(X^0) = 0.6091\}; \quad (152)$$

$\lambda^0 = 0.6091$ is the maximum lower level among all relative estimates measured in the relative units: $\lambda^0 = \min(\lambda_1(X^0), \lambda_2(X^0), \lambda_3(X^0), \lambda_4(X^0)) = 0.6091$.

$\lambda^0 = 0.6091$ also call the guaranteed result in the relative units, i.e. $\lambda_k(X^0), k = \overline{1, K}$ and according to the characteristic of the technical system $f_k(X^0), k = \overline{1, K}$ it is impossible to improve, without worsening at the same time other characteristics.

Let's notice that according to the theorem 1, in X^0 point criteria 2, 3, 4: $\lambda_1(X^0), \lambda_3(X^0), \lambda_4(X^0)$ are contradictory. This contradiction is defined by equality of

$$\lambda^0 = \min(\lambda_1(X^0), \lambda_2(X^0), \lambda_3(X^0), \lambda_4(X^0)), \lambda^0 = 0.6091, \text{ and other criteria inequality } \{\lambda_k(X^0)\} > \lambda^0.$$

Theorem 1 serves as the basis for determining the correctness of the solution of a vector problem. As a rule, the equation for two criteria is fulfilled for the two criteria:

$$\lambda^0 = \lambda_q(X^0) = \lambda_p(X^0), q, p \in K, X \in S$$



(in our example, such criteria are 2, 3, 4), for other criteria is defined as inequality.

3.2.7. Stage 6. Geometric interpretation of the results of the VPMP solution with 4 parameters and 4 criteria into a two-dimensional coordinate system in relative units.

For a geometric interpretation of the results of the VPMP solution with 4 parameters and 4 criteria, we will introduce changes in the two-dimensional coordinate system (with 2 parameters) in relative units. In VPMP (140)-(145) the parameters are x_1 and x_3 are considered as variables, parameters y_2 and y_4 are considered permanent. In accordance with the outcome of the decision of VPMP (140)-(145), with equivalent criteria:

$$X^o = \{X^o, \lambda^o\} = \{x_1 = 45.4458, x_2 = 49.99, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}.$$

Let's assign a dimension to the constant parameters: $x_2 = 49.99, x_4 = 2.2$.

As a result, VPMP (140)-(145) became two-dimensional.

Results of the solution of VPMP (140)-(145) with two variables x_1 and x_3 . In the designation of the results of the solution, an additional letter "o" $X1o_{max}$ was introduced.

1. According to the first criterion, the coordinates of the point by the maximum are equal to:

$$X1o_{max} = \{x_1 = 47.6032 \quad x_2 = 49.9964 \quad x_3 = 8.0000 \quad x_4 = 2.2000\}.$$

Values of the four criteria at the point $X1o_{max}$: $FX1o_{max} = \{f_1(X1o_{max}) = 407.9$
 $f_2(X1o_{max}) = 1401.5 \quad f_3(X1o_{max}) = 190.5 \quad f_4(X1o_{max}) = 74.4$.

Values are relative to the estimates of the criteria at the point $X1o_{max}$:

$$LX1o_{max} = \{\lambda_1(X1o_{max}) = 1.1666 \quad \lambda_2(X1o_{max}) = 0.9406$$

$$\lambda_3(X1o_{max}) = 0.5046 \quad \lambda_4(X1o_{max}) = 0.0192.$$

Coordinates of the point according to the first criterion for the minimum:

$$X1o_{min} = \{33.9000 \quad 49.9964 \quad 2.100 \quad 2.2000\}.$$

Values of the four criteria at the point $X1o_{min}$:

$$FX1o_{min} = \{1.0e+03 * 0.3100 \quad 2.1799 \quad 0.1966 \quad 0.0260\}.$$

Values are relative to the estimates of the criteria at the point $Y1o_{min}$:

$$LX1o_{min} = \{0.0806 \quad 0.2200 \quad 0.6505 \quad 1.1562\}.$$

2. According to the second criterion, the coordinates of the point, functions and relative estimates for the maximum and minimum are equal:

$$X2o_{max} = \{56.6036 \quad 49.9964 \quad 8.0000 \quad 2.2000\}.$$

$$FX2o_{max} = \{1.0e+03 * 0.4368 \quad 1.1630 \quad 0.1914 \quad 0.0858\}.$$

$$LX2o_{max} = \{1.4871 \quad 1.1615 \quad 0.5205 \quad -0.2501\}.$$

$$X2o_{min} = \{33.9000 \quad 49.9964 \quad 2.1000 \quad 2.2000\}.$$

$$FX2o_{min} = \{1.0e+03 * 0.3100 \quad 2.1799 \quad 0.1966 \quad 0.0260\}.$$

$$LX2o_{min} = \{0.0806 \quad 0.2200 \quad 0.6505 \quad 1.1562\}.$$

3. According to the third criterion, the coordinates of the point, functions and relative estimates for maximum and minimum:

$$X3o_{max} = \{33.9000 \quad 49.9964 \quad 2.1000 \quad 2.2000\}.$$

$$FX3o_{max} = \{1.0e+03 * 0.3100 \quad 2.1799 \quad 0.1966 \quad 0.0260\}.$$

$$LX3o_{max} = \{0.0806 \quad 0.2200 \quad 0.6505 \quad 1.1562\}.$$

$$X3o_{min} = \{78.9999 \quad 49.9964 \quad 8.0000 \quad 2.2000\}.$$

$$FX3o_{min} = \{518.7746 \quad 602.8613 \quad 192.6992 \quad 112.8235\}.$$

$$LX3o_{min} = \{2.3963 \quad 1.6800 \quad 0.5536 \quad -0.8846\}.$$

4. According to the fourth criterion, the coordinates of the point, functions and relative estimates for maximum and minimum are equal:

$$X4o_{max} = \{38.7000 \quad 49.9964 \quad 2.1000 \quad 2.2000\}.$$

$$FX4o_{max} = \{1.0e+03 * 0.3250 \quad 2.1021 \quad 0.1972 \quad 0.0301\}.$$

$$LX4o_{max} = \{0.2471 \quad 0.2920 \quad 0.6662 \quad 1.0614\}.$$

$$X4o_{min} = \{63.4651 \quad 49.9964 \quad 8.0000 \quad 2.2000 \quad 23.1979\}.$$



$FX_{4min} = \{460.4076 \ 986.2753 \ 191.8170 \ 94.3384\}.$

$LX_{4min} = \{1.7488 \ 1.3251 \ 0.5316 \ -0.4499\}.$

Let us present the results of the solution as a whole with two variable parameters x_1 and x_3 (**two-dimensional problem**):

$$\begin{aligned} X_{opt}(1,:) &= \{47.6032 \ 44.1968 \ 8.0000 \ 2.2000\}, & \lambda_1(X_{1max}) &= \mathbf{1.1666}; \\ X_{opt}(2,:) &= \{56.6036 \ 35.1964 \ 8.0000 \ 2.2000\}, & \lambda_2(X_{2max}) &= \mathbf{1.1615}; \\ X_{opt}(3,:) &= \{32.9000 \ 60.0000 \ 2.1000 \ 7.0000\}, & \lambda_3(X_{3max}) &= \mathbf{0.6505}; \\ X_{opt}(4,:) &= \{37.7000 \ 60.0000 \ 2.1000 \ 2.2000\}, & \lambda_4(X_{4max}) &= \mathbf{1.0614}; \\ X_{o(1:4)} &= \{44.7082 \ 50.7315 \ 4.3603 \ 2.2000\}, & \lambda(X_o) = \lambda^o &= \mathbf{0.6081}. \end{aligned} \quad (153)$$

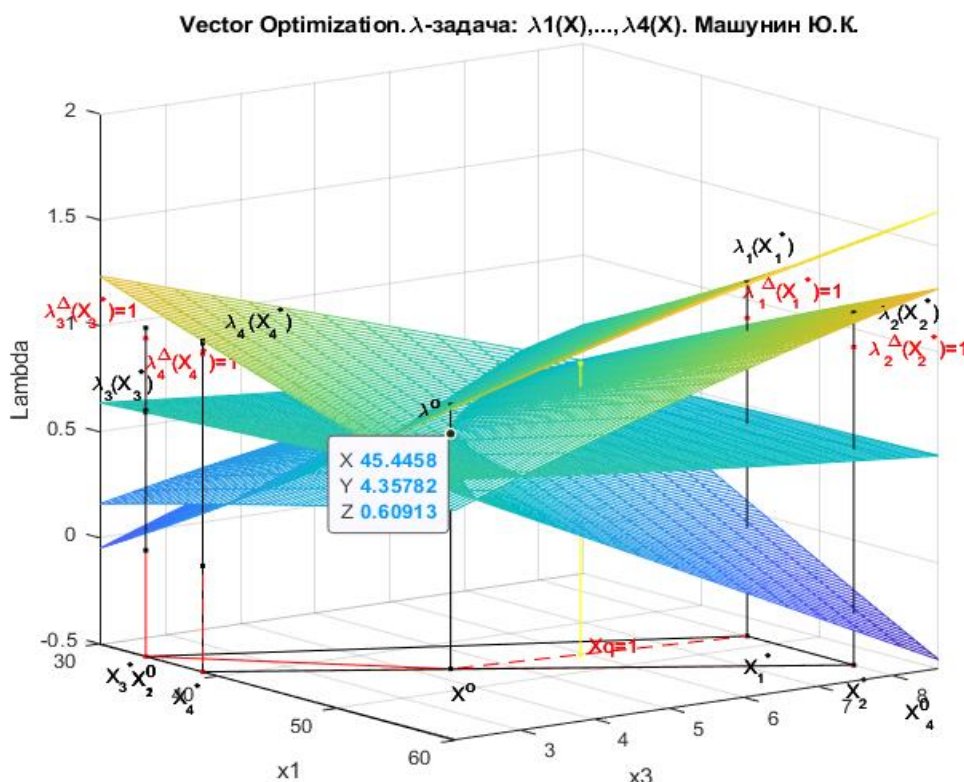


Figure 12. Geometric interpretation of four functions in relative units $\lambda_1(X_o), \lambda_2(X_o), \lambda_3(X_o), \lambda_4(X_o)$ in the three-dimensional coordinate system $x_1 \ x_2$ and λ

Characteristics of the technical system in relative units $\lambda_1(X_o), \lambda_2(X_o), \lambda_3(X_o), \lambda_4(X_o)$ are shown in Fig. 12 in three-dimensional space $x_1 \ x_3$ and λ , where the third axis λ is the relative estimate. In the admissible set of points S formed by constraints (144)-(145), the optimum points are $X_1^*, X_2^*, X_3^*, X_4^*$, combined into a contour, represent a set of Pareto-optimal points, $S^o \subset S$ shown in Fig. 11. The coordinates of these points, as well as the characteristics of the technical system in relative units $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$ are shown in Fig. 12 in three-dimensional space $x_1 \ x_2$ and λ , where the third axis λ is a relative estimate.

3.2.8. Stage 7. Results of solving the Vector Problem of Mathematical Programming with equivalent criteria - preparation of information for decision-making with the priority of the criterion. As a result, the decision of the VPMP with equivalent criteria was received.

Results of solving a vector problem of mathematical programming.



(Mathematical model of the Engineering System (ES) *with equivalent criteria*).

1. **Criteria (characteristics of ES), parameters at the optimum point:**

Criteria 1: $f_1^* = f_1(X_1^*) = -392$; **Parameters:** $X_1^* = \{x_1 = 47.6, x_2 = 44, x_3 = 8., x_4 = 2.2\}$;

Criteria 2: $f_2^* = f_2(X_2^*) = 1337$; **Parameters:** $X_2^* = \{x_1 = 56.6, x_2 = 35.19, x_3 = 8, x_4 = 2.2\}$; (154)

Criteria 3: $f_3^* = f_3(X_3^*) = -210.5$; **Parameters:** $X_3^* = \{x_1 = 32.9, x_2 = 59, x_3 = 2.1, x_4 = 7\}$;

Criteria 4: $f_4^* = f_4(X_4^*) = 32.68$; **Parameters:** $X_4^* = \{x_1 = 37.7, x_2 = 60, x_3 = 2.1, x_4 = 2.2\}$.

2. **Criteria (characteristics of ES), parameters at the anti optimum point:**

Criteria 1: $f_1^0 = f_1(X_1^0) = 302.7$ **Parameters:** $X_1^0 = \{x_1 = 31.9, x_2 = 59, x_3 = 2.1, x_4 = 7\}$;

Criteria 2: $f_2^0 = f_2(X_2^0) = -2417$; **Parameters:** $X_2^0 = \{x_1 = 32.9, x_2 = 59, x_3 = 2.1, x_4 = 7\}$; (155)

Criteria 3: $f_3^0 = f_3(X_3^0) = 170$; **Parameters:** $X_3^0 = \{x_1 = 78.9, x_2 = 10.4, x_3 = 8, x_4 = 4.5\}$;

Criteria 4: $f_4^0 = f_4(X_4^0) = -75.2$; **Parameters:** $X_4^0 = \{63.465, x_2 = 24.5, x_3 = 8, x_4 = 6\}$;

3. **Criteria (characteristics of ES), parameters at the optimum point** in relative units at the point X^0 : $\lambda^0 = 0.6091$ – maximum lower level in relative units.

$\lambda_k(X^0), k = \overline{1, K}$ - values of criteria (characteristics) in physical units:

$\{\lambda_1(X^0) = 0.7452, \lambda_2(X^0) = 0.6091, \lambda_3(X^0) = 0.6091, \lambda_4(X^0) = 0.6091\}$; (156)

Parameters: $X^0 = \{X^0 = \{x_1 = 45.4, x_2 = 49.99, x_3 = 4.357, x_4 = 2.2, \lambda^0 = 0.6091\}$. (157)

4. **Criteria (characteristics of ES) in physical units:**

$f_k(X^0), k = \overline{1, K}$ - values of criteria (characteristics) in physical units: (158)

$\{f_1(X^0) = 369.9, f_2(X^0) = 1759.6, f_3(X^0) = 194.9, f_4(X^0) = 49.3\}$;

5. **Engineering System Information in Physical Units for Criterion Priority Decision Making:**

1. $f_1(X) \rightarrow \max$: $f_1^* = f_1(X_1^*) = -392.9 \leq f_1(X^0) = 369.9 \leq f_1^0 = f_1(X_1^0) = 302.76$;

2. $f_2(X) \rightarrow \min$: $f_2^* = f_2(X_2^*) = 1337 \leq f_2(X^0) = 1758.6 \leq f_2^0 = f_2(X_2^0) = -2417.5$; (159)

3. $f_3(X) \rightarrow \max$: $f_3^* = f_3(X_3^*) = -210 \leq f_3(X^0) = 194.9 \leq f_3^0 = f_3(X_3^0) = 170.53$;

4. $f_4(X) \rightarrow \min$: $f_4^* = f_4(X_4^*) = 32.68 \leq f_4(X^0) = 49.3 \leq f_4^0 = f_4(X_4^0) = -75.205$.

3.2.9. **Conclusion on the Results of Solving a Vector Problem of Mathematical Programming with Equivalent Criteria.**

We presented a methodology for solving a vector problem of mathematical programming with equivalent criteria (characteristics) on the example of an engineering system - the technical system. In the process of solving the problem, we presented the MATLAB system with a geometric interpretation of the solution results in a two-dimensional coordinate system. The results of solving a vector problem of mathematical programming with equivalent criteria with four variables coincide with the results of solving a vector problem with two variables.

Thus, for the first time in domestic and foreign practice, the transition and its geometric illustration from the N -dimensional to the two-dimensional dimension of the function in vector problems of mathematical programming with the corresponding errors of linear approximation are shown. We have prepared information for the study and selection of the optimal solution with the priority of one or another criterion (characteristic) of the engineering system. The next part of the work is aimed at solving this problem.

3.3. **Part 2. Selection of optimal parameters of engineering systems under uncertainty - with the priority of the criterion based on multidimensional mathematics.**

Part 2 includes two blocks: **Block 3.** Analysis, comparison of the results of the solution of the VZMP with equivalent criteria of a complex engineering system (model of the technical system) and preparation for the selection of optimal parameters according to the priority criterion for solving the vector problem (Section 3.3).

Block 4. Study and selection of optimal parameters of the technical system according to priority criteria k_1, k_2, k_3, k_4 and geometric interpretation of the solution results in relative and physical units based on multidimensional optimization (Section 3.4).



3.3.1. Block 3. Analysis, comparison of the results of the VPMP solution with equivalent criteria of a complex engineering system (technical system model) and preparation for the selection of optimal parameters according to the priority criterion.

Let us compare the results of solving a vector problem of mathematical programming with equivalent criteria (140)-(145) with four variables x_1, x_2, x_3 and x_4 with the results of solving the vector problem (140)-(145) with two variables x_1 and x_3 .

Results of solving a four-dimensional VPMP (140)-(145) with variable coordinates $\{x_1, x_2, x_3, x_4\}$ are presented in (150), (151), (152), and the results of solving a two-dimensional VPMP with variable coordinates $\{x_1, x_3\}$ are presented in (153). The results of solving and comparing the studied problems are presented in a three-dimensional coordinate system x_1, x_3, λ . (Designation: x_1, x_3 and λ in Fig. 13).

$X^0 = \{X^0, \lambda^0\} = \{X^0 = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^0 = 0.6091\}$
of (150); and relative assessment:
 $\{\lambda_1(Y^0) = 0.7452, \lambda_2(Y^0) = 0.6091, \lambda_3(Y^0) = 0.6091, \lambda_4(Y^0) = 0.6091\}$
from (151) in the same coordinates x_1, x_3 .

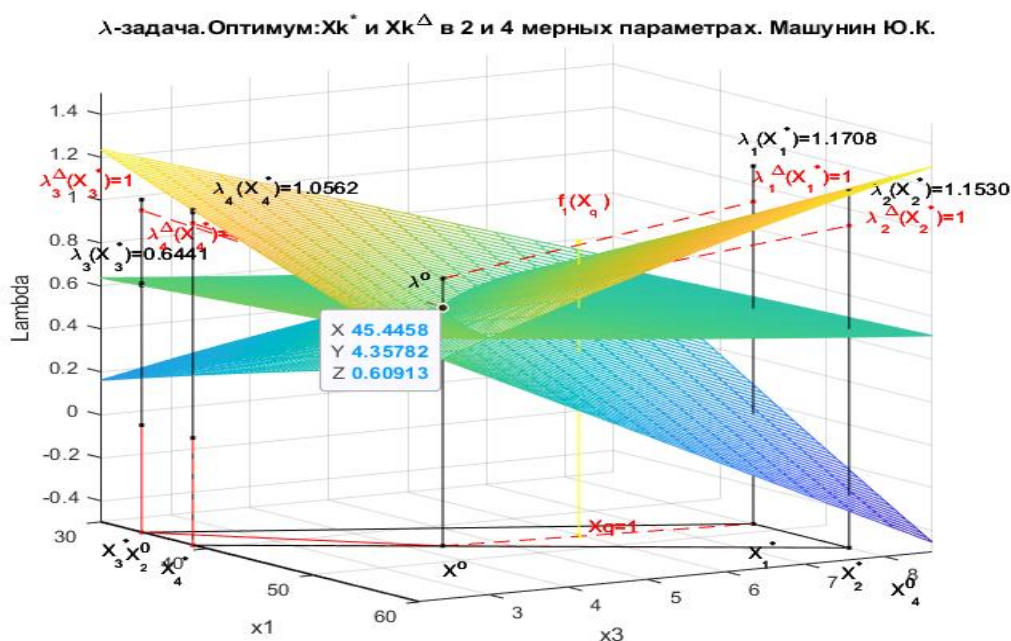


Figure 13. Solving λ -problems in a three-dimensional coordinate system x_1, x_3 and λ . Consider, for example, the optimal point X_3^* . Function, $\lambda_3(X)$ is formed from the function $f_3(X)$ with variable coordinates $\{x_1, x_3\}$, with constant coordinates $\{x_2=49.54, x_4=2.2\}$, taken from the optimal point X^0 (150). At the point X_3^* . The relative estimate is $\lambda_3(X_3^*) = 0.6441$ – shown in Fig. 13 with a black dot

But we know that the relative estimate of $\lambda_3(X_3^*)$ obtained from the function $f_3(X_3^*)$. In the third step, it is equal to one, let's denote it as $\lambda_3^A(X_3^*) = 1$ – shown in Fig. 13 with a red dot.

In summary: First, with equivalent criteria for the coordinates of the optimum points x_1, x_3 :

$X^0 = \{X^0, \lambda^0\} = \{X^0 = \{x_1 = 45.44, x_3 = 4.3578, \lambda^0 = 0.6091\}$ - in a two-dimensional

$X^0 = \{X^0, \lambda^0\} = \{X^0 = \{x_1 = 45.44, x_2 = 49.54, x_3 = 4.3578, x_4 = 2.2, \lambda^0 = 0.6091\}$

in a four-dimensional system they coincide (using the software capabilities of the MATLAB system, we will show the numerical values of the optimum point $\{X^0, \lambda^0\}$ in Fig. 13);



secondly, at the optimum point X_3^* optimal values of the criteria $f_k(X_k^*)$, $k = 3$ and relative estimates $\lambda_k(X_k^*) = 0.6441$, $k = 3$ and $\lambda_k^\Delta(X_k^*) = 1$, $k = 3$ do not match.

The difference between the relative estimates is $\lambda_3^\Delta(X_3^*)=1$ and $\lambda_3(X_3^*) = 0.6441$ is an error $\Delta=0.3559$ in the transition from a four-dimensional (and in the general case N -dimensional) to a two-dimensional system of measurements. Similarly, in Fig. 13 shown:

point X_1^* , corresponding relative estimates $\lambda_1(X_1^*) = 1.1708$, $\lambda_1^\Delta(X_1^*) = 1$;

point X_2^* , corresponding relative estimates $\lambda_2(X_2^*) = 1.1530$, $\lambda_2^\Delta(X_2^*) = 1$;

point X_3^* , corresponding relative estimates $\lambda_3(X_3^*) = 0.6441$, $\lambda_3^\Delta(X_3^*) = 1$;

point X_4^* , corresponding relative estimates $\lambda_4(X_4^*) = 1.0562$, $\lambda_4^\Delta(X_4^*) = 1$.

Conclusion from the analysis: Information about the Engineering System in physical units for decision-making with the priority of the criterion.

Based on the results of solving a vector problem of mathematical programming with equivalent criteria, we formulated information for improving and making a management decision for any criterion (characteristics of an engineering system):

$$f_k(X_k^*) \leq f_k(X^0) \leq f_k(X_k^0), k = \overline{1, K}.$$

Information about the Engineering System in physical units for decision-making:

1. $f_1(X) \rightarrow \max$: $f_1^* = f_1(X_1^*) = -392.9 \leq f_1(X^0) = 369.9 \leq f_1^0 = f_1(X_1^0) = 302.7645$;
2. $f_2(X) \rightarrow \min$: $f_2^* = f_2(X_2^*) = 1337.38 \leq f_2(X^0) = 1759.6 \leq f_2^0 = f_2(X_2^0) = -2452.3$;
3. $f_3(X) \rightarrow \max$: $f_3^* = f_3(X_3^*) = -210.57 \leq f_3(X^0) = 194.9 \leq f_3^0 = f_3(X_3^0) = 170.53$;
4. $f_4(X) \rightarrow \min$: $f_4^* = f_4(X_4^*) = 32.68 \leq f_4(X^0) = 49.3 \leq f_4^0 = f_4(X_4^0) = -75.205$.

3.3.2. Stage 9. Preparation of information for the selection of optimal parameters for any priority criterion and geometric interpretation of the solution results

When preparing information for the selection of optimal parameters according to the priority criterion and geometric interpretation of the results of solving a vector problem, we use Fig. 13, from which the functions $\{\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)\}$ in relative units, and represent it in Fig. 14.

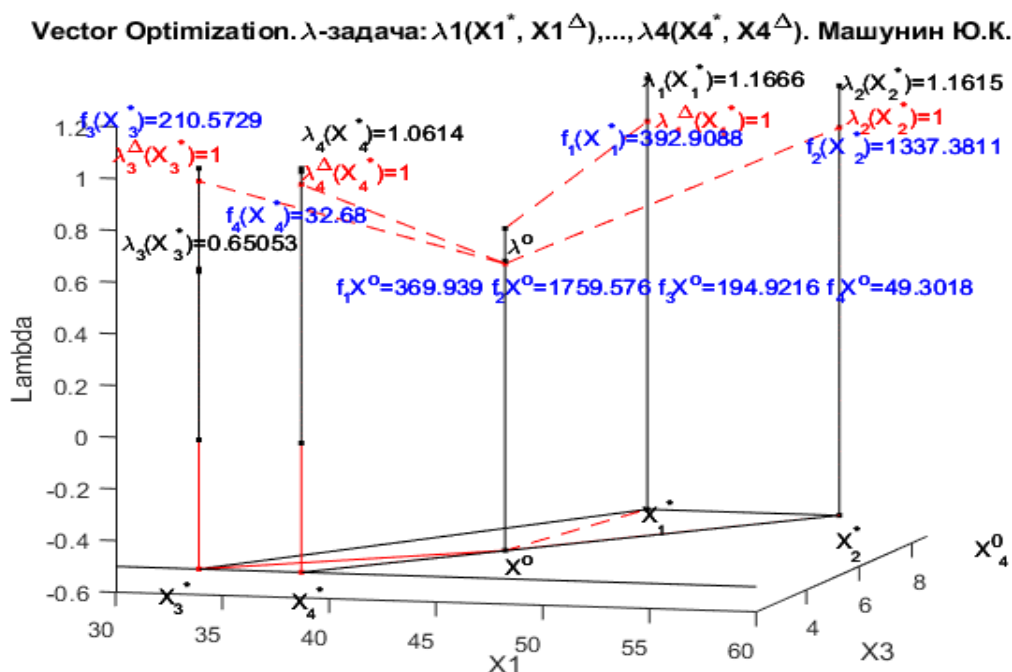


Figure 14. λ -problem in the coordinate system x_1 x_3 and λ . Geometric interpretation of the results of solving four criteria in relative and physical units



1. Results of the solution $\lambda_1(X_1^*), \dots, \lambda_4(X_4^*)$ in a two-dimensional coordinate system $x_1 x_3$ and λ . 2. Theoretical results of the solution $\lambda_1^\Delta(X_1^*) = 1, \dots, \lambda_4^\Delta(X_4^*) = 1$ in a two-dimensional coordinate system $x_1 x_3$ and λ . (Red color). 3. Results of solution $f_1(X_1^*) = \dots = f_4(X_4^*)$ in physical units, corresponding to $\lambda_1^\Delta(X_1^*) = 1, \dots, \lambda_4^\Delta(X_4^*) = 1$ – relative units. 4. Results of the solution $f_1(X_1^0), \dots, f_4(X_4^0)$ in physical units with equivalent criteria. (Purple)

Let's imagine in Fig. 14 Information data on relative estimates $\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)$ and physical data $f_1(X), f_2(X), f_3(X), f_4(X)$ obtained when solving a vector problem with equivalent criteria.

1. Let's present the results of the solution $\lambda_1(X_1^*), \dots, \lambda_4(X_4^*)$, which are obtained in the two-dimensional coordinate system $x_1 x_3$ and λ (see section 5.1.7):

point X_1^* , corresponding relative estimate $\lambda_1(X_1^*) = 1.1708$;

point X_2^* , corresponding relative estimate $\lambda_2(X_2^*) = 1.1530$;

point X_3^* , corresponding relative estimate $\lambda_3(X_3^*) = 0.6441$;

point X_4^* , corresponding relative estimate $\lambda_4(X_4^*) = 1.0562$.

2. Theoretical results of the solution $\lambda_k^\Delta(X_k^*) = 1, k = \overline{1, K}, K = 4$ in the two-dimensional coordinate system $x_1 x_3$ and λ . (Red):

point X_1^* , corresponding relative estimate $\lambda_1^\Delta(X_1^*) = 1$;

point X_2^* , corresponding relative estimate $\lambda_2^\Delta(X_2^*) = 1$;

point X_3^* , corresponding relative estimate $\lambda_3^\Delta(X_3^*) = 1$;

point X_4^* , corresponding relative estimate $\lambda_4^\Delta(X_4^*) = 1$.

3. Results of solution $f_1(X_1^*) = \dots = f_4(X_4^*)$ in physical units, corresponding to $\lambda_1^\Delta(X_1^*) = 1, \dots, \lambda_4^\Delta(X_4^*) = 1$ – relative units that are obtained at the third step of the VPMP solution with equivalent criteria in (146) (Purple):

the point X_1^* , the corresponding physical quantity $f_1(X_1^*) = 392.908$;

point X_2^* , corresponding physical quantity $f_2(X_2^*) = 1337.38$;

point X_3^* , corresponding physical quantity $f_3(X_3^*) = 210.57$;

point X_4^* , corresponding physical quantity $f_4(X_4^*) = 32.68$.

4. Results of solving all criteria (characteristics of the technical system):

$f_1(X_1^0) = \dots = f_4(X_4^0)$ in physical units at the optimum point $X_k^0, k = \overline{1, K}$ with equivalent criteria (Violet): $f_1 X_1^0 = 369.93, f_2 X_2^0 = 1759.57, f_3 X_3^0 = 194.92, f_4 X_4^0 = 49.3018$.

Let's analyze the results obtained. A linear function connecting the points $\lambda^0(X^0) = 0.6091$ and $\lambda_2^\Delta(X_2^*) = 1$ in relative units characterizes the function $f_2(X)$ in relative units in the four-dimensional dimension of the parameters $x_1, \dots, x_4$. This linear function represents the geometric interpolation of the function $f_2(X)$ in relative units from N -dimensional (4-dimensional) to two-dimensional coordinate system. As can be seen from Fig. 14 Corresponding physical quantities $f_2(X)$ lie within the limits of:

$$f_2(X^0) = 1759.576 \leq f_2(X) \leq f_2(X_2^*) = 1337.38.$$

Note that the function $f_2(X)$ minimized—at the optimum point of X_2^* . The value is minimal. Similar $f_2(X)$ ratios are constructed in Fig. 14 for each criterion. Relations $f_2(X)$ are the basis for the study of the selected criterion in the sequence:

First, to select the value of the criterion in physical units:

secondly, the calculation of the corresponding relative valuation;

thirdly, the calculation of the relevant parameters of the technical system;

fourth, making a final decision.

The paper shows the choice of optimal parameters of the technical system for a given value of each criterion $k = \overline{1, K}, K = 4$.

The value of 1 of the criteria is given. $f_1(X) = 380$, which lies within: (160)



$$f_1(X^0) = 369.9 \geq f_1(X) = \mathbf{380} \geq f_1(X_1^*) = -392.9. \text{ (Block 4.k1)}$$

The value of criterion 2 is $f_2(X) = 1500$, which lies within:

$$f_2(X^0) = -1759.6 \leq f_2(X) = \mathbf{1500} \leq f_2(X_2^*) = -1337.38. \text{ (Block 4.k2)}$$

The value of 3 of the criteria is $f_3(X) = 200$, which lies within:

$$f_3(X^0) = 194.9 \leq f_3(X) = \mathbf{200} \leq f_3(X_1^*) = -210.57. \text{ (Block 4.k3)}$$

The value of 3 of the criteria is $f_4(X_1^0) = 40$, which lies within:

$$f_4(X^0) = -49.3 \geq f_4(X) = \mathbf{40} \geq f_4(X_1^*) = -32.68. \text{ (Block 4.k4).}$$

The implementation of the selection of optimal parameters of the technical system for a given value of each criterion $f_k(X)$, $k = \overline{1, K}$, $K = 4$ is presented in **block 4**.

3.3.3. Block 4. Study and selection of optimal parameters of the technical system according to priority criteria k1, k2, k3, k4 and geometric interpretation of the solution results in relative and physical units based on multidimensional optimization.

The selection of the optimal parameters of a complex engineering (technical) system and the geometric interpretation of the solution results according to the priority criterion corresponds to the number of criteria $K=4$: Block **k1**, Block **k2**, Block **k3**, Block **k4**.

Name of the block.

Block **k1**, ..., Block **k4**. **Solution of VPMP** - a model of a complex engineering system with a given priority of the first criterion in multidimensional mathematics.

Each parameter selection block for the corresponding criterion includes the following sections.

1.k1, ..., 1.k4. Solution of a vector problem - a model of a complex engineering system (technical system) with a given priority of the first criterion, including:

Solving a vector problem of mathematical programming - a model of an engineering system with equivalent criteria (Step 1).

Select the priority criterion $q \in K$ (Step 2).

Select the numerical value of the selected priority criterion $q(K)$ (Steps 3, 4).

Solutions to a vector problem of mathematical programming - a model of an engineering system with a given priority of the $q \in K$ criterion (Steps 5–8).

2.k1, ..., 2.k4. Analysis of the results of solving a vector problem of mathematical programming - a model of an engineering system at a given priority of the first criterion.

3.k1, ..., 3.k4. Geometric interpretation of the results of solving the vector problem of choosing optimal parameters by the priority criterion in relative units.

4.k1, ..., 4.k4. Geometric interpretation of the results of the VPMP solution (functions: $f_1(X), \dots, f_4(X)$) with the priority of the first criterion – the model of the technical system when designing in a three-dimensional coordinate system **in physical units**.

1. Geometric interpretation of the results of the solution of **the first criterion** with the priority of the first criterion – the model of the engineering system when designing in a three-dimensional coordinate system **in physical units**. ...

4. Geometric interpretation of the results of the solution of the VPMP of **the fourth criterion** with the priority of the first criterion - the model of the engineering system when designing in a three-dimensional coordinate system **in physical units**.

3.3.4. Block k1. Selection of optimal parameters of the engineering system (technical system) according to the first priority criterion, geometric interpretation of the results of the VPMP solution.

1.k1. Solution of VPMP - a model of a complex engineering system with a given priority of the first criterion in multidimensional mathematics.

The person making decisions, as a rule, is the designer of the technical system.

Step 1. The solution of a VPMP with equivalent criteria.

Results of the decision are presented in Stage 4.



Pareto-optimal points $S^o \subset S$ is between the optimal points $X_1^* X_{13}^o X_3^* X_{34}^o X_4^* X_{42}^o X_2^* X_{21}^o X_1^*$. Let us analyze the set of Pareto points $S^o \subset S$. For this purpose, we will connect the auxiliary points: $X_{13}^o X_{34}^o X_{42}^o X_{21}^o$ with the point X^o , which conventionally represents the center of the Pareto set. As a result, the solution obtained four subsets of points $X \in S_q^o \subset S^o \subset S, q = \overline{1,4}$. A subset of points точек $S_1^o \subset S^o \subset S$ – points $X_1^* X_{13}^o X^o X_{12}^o X_1^*$ is characterized by the fact that the relative score $\lambda_1 \geq \lambda_2, \lambda_3, \lambda_4$, i.e., in the field of the first criterion S_1^o takes precedence over the others. Similarly, S_2^o, S_3^o, S_4^o - subsets of points, is characterized by the fact that the second, third, and fourth criteria take precedence over the others, respectively.

The set of Pareto-optimal points is denoted by $S^o = S_1^o \cup S_2^o \cup S_3^o \cup S_4^o$. The geometrical interpretation of the coordinates of all the obtained points and the relative estimates is presented in two-dimensional space $\{x_1 x_3\}$ in Figure 11. These coordinates are shown in three measured spaces $\{x_1 x_3 \lambda\}$ in Figure 12, where the third axis λ is the relative valuation.

Limitations in Figure 12 lowered to -0.5 (to make the restrictions visible). The obtained information is the basis for further study of the structure of the Pareto set.

If results of the solution of a VPMP with equivalent criteria do not satisfy the person making the decision, then the choice of an optimal solution is carried out from any subset of points of $S_1^o, S_2^o, S_3^o, S_4^o$. These subsets of Pareto points are shown in Figure 12 as functions $f_1(X), f_2(X), f_3(X), f_4(X)$.

Step 2. Choice of priority criterion of $q \in K$.

From the theory it is known that in an optimum point of X^o there are always two most contradictory criteria: $q \in K$ and $v \in K$ for which in the relative unit's precise equality is carried out:

$$\lambda^o = \lambda_q(X^o) = \lambda_p(X^o), q, p \in K, X \in S,$$

and for the others it is carried out inequalities: $\lambda^o \leq \lambda_k(X^o), \forall k \in K, q \neq p \neq k$.

In the technical system model (140)-(145) and the corresponding λ -problem (149), these criteria are the second and third: $\lambda^o = \lambda_2(X^o) = \lambda_3(X^o) = 0.6091$.

As a rule, from a pair of conflicting criteria $\lambda^o = \lambda_1(X^o) = \lambda_3(X^o) = 0.6091$, a criterion chosen by the decision maker would be improved. Such criterion is called "priority criterion", we will designate it $q = 1 \in K$.

This criterion is investigated in interaction with the first criterion of $q = 1 \in K$.

Selection of a priority criterion: The display provides general information for decision-making: Criteria at the optimum point X^o :

$$f_k(X^o) = \{f_1(X^o) = 369.9, f_2(X^o) = 1759.6, f_3(X^o) = 194.9, f_4(X^o) = 49.3\}. \quad (161)$$

Relative estimates in X^o :

$$\lambda_k(X^o) = \{\lambda_1(X^o) = \mathbf{0.7452}, \lambda_2(X^o) = \mathbf{0.6091}, \lambda_3(X^o) = \mathbf{0.6091}, \lambda_4(X^o) = \mathbf{0.6091}\} \quad (162)$$

$$f_1^0 = f_1(X_1^0) = 302.7645; f_2^0 = f_2(X_2^0) = -2417.52; f_3^0 = f_3(X_3^0) = 170.53; f_4^0 = f_4(X_4^0) = -75.2.$$

Conclusion: Criteria 2, 3, 4 are the most contradictory, of which we choose the priority one.

This section examines **the first criterion** (function) $q = 1 \in K$.

On the display the message is given:

$q = \text{input ('Enter priority criterion (number) of } q = \text{'}) - \text{Entered: } q = 1$.

Step 3. Numerical limits of change in the value of the priority of the criterion $q = 1 \in K$ are formed. For the priority criterion $q = 1 \in K$ changes in the numerical limits in natural units are determined during the transition from the point of the optimum X^o (136) to the point X_q^* , obtained in the first step. Information on criterion $q=1$ is displayed on:

$$f_q(X^o) = 369.9 \leq f_q(X) \leq 392.9 = f_q(X_q^*), q = 1 \in K. \quad (163)$$

In the relative units the criterion of $q=1$ changes in the following limits:

$$\lambda_q(X^o) = 0.7452 \leq \lambda_q(X) \leq 1 = \lambda_q(X_q^*), q = 1 \in K.$$

These data it is analysed.



Step 4. Choice of size of priority criterion of $q \in K$. (Decision-making).

On the message: "Enter the size of priority criterion f_q " - we enter, the size of the characteristic defining structure of material: $f_q = 380$.

Step 5. The relative assessment is calculated.

For the chosen size of priority criterion $f_q = 380$ the relative assessment is calculated:

$$\lambda_q = \frac{f_q - f_q^0}{f_q^* - f_q^0} = \frac{380 - 369.93}{392.9 - 369.93} = 0.8568,$$

which upon transition from X^0 point to X_3^* lies in limits:

$$\lambda_q(X^0) = 0.7372 \leq \lambda_q(X) = 0.8568 \leq 1.0 = \lambda_q(X_q^*), q = 1 \in K. \quad (164)$$

Step 6. Let's calculate coefficient of the linear approximation

Assuming the linear nature of change of criterion of $f_q(X)$ in (139) and according to the relative assessment of λ_q in (140), using reference methods of the linear approximation, we will calculate a constant of proportionality between $\lambda_q(X^0)$, λ_q which we will call ρ :

$$\rho = \frac{\lambda_q - \lambda_q(X^0)}{\lambda_q(X_q^*) - \lambda_q(X^0)} = \frac{0.8568 - 0.6091}{1 - 0.6091} = 0.4380, q = 1 \in K. \quad (165)$$

Step 7. Let's calculate coordinates of a priority of criteria with dimension of f_q

Assuming the linear nature of change of a vector of

$X^q = \{x_1 \ x_2 \ x_3 \ x_4\}$, $q = 1$ we will determine point coordinates with dimension of $f_q = 380$, the relative assessment (152):

$$\begin{aligned} x_{\lambda=0.74}^{q=3} &= \{x_1 = X^0(1) + \rho(X_q^*(1) - X^0(1)), \\ x_2 &= X^0(2) + \rho(X_q^*(2) - X^0(2)), \\ x_3 &= X^0(3) + \rho(X_q^*(3) - X^0(3)), \\ x_4 &= X^0(4) + \rho(X_q^*(4) - X^0(4))\}, \end{aligned} \quad (166)$$

where $\{X^0, \lambda^0\} = \{X^0 = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^0 = 0.6091\}$.

$X_1^* = \{x_1 = 47.6032, x_2 = 44.1968, x_3 = 8.0, x_4 = 2.2\}$.

As result of the decision (166) we will receive X^q point with coordinates:

$$X^q = \{x_1 = 46.3908, x_2 = 47.4561, x_3 = 5.9531, x_4 = 2.2000\}. \quad (167)$$

Step 8. Calculation of **the main indexes of a point of X^q** .

For the received X^q point, we will calculate:

all criteria in physical units $f_k(X^q) = \{f_k(X^q), k = \overline{1, K}\}$,

$$f_k(X^{q=1}) = \{f_1(X^q) = 381.5, f_2(X^q) = 1597.9, f_3(X^q) = 190.4, f_4(X^q) = 58.6\}; \quad (168)$$

all relative estimates of criteria:

$$\lambda^q = \{\lambda_k^q, k = \overline{1, K}\}, \lambda_k(X^q) = \frac{f_k(X^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}, \quad (169)$$

$$\lambda_k(X^q) = \{\lambda_1(X^q) = 0.8733, \lambda_2(X^q) = 0.7588, \lambda_3(X^q) = 0.4953, \lambda_4(X^q) = 0.3896\};$$

Minimum relative estimates of criteria: $\min \lambda(X^q) = \min_{k \in K} (\lambda_k(X^q)) = 0.3896$.

$$P^q = [p_1^1 = 1.0, p_2^1 = 1.1508, p_3^1 = 1.7632, p_4^1 = 2.2416];$$

vector priority criterion $P^q(X)$: $P^q(X) = \{p_k^q = \frac{\lambda_k(X^q)}{\lambda_k(X^q)}, k = \overline{1, K}\}$:

$$\lambda_k(X^q) * P^q = \{p_1^1 * \lambda_1(X^q) = 0.8733, p_2^1 * \lambda_2(X^q) = 0.8733, p_3^1 * \lambda_3(X^q) = 0.8733, p_4^1 * \lambda_4(X^q) = 0.8733\}.$$

Minimum relative estimates of criteria $q \in K$:

$$\lambda^{oo} = \min(p_1^3 \lambda_1(X^q), p_2^3 \lambda_2(X^q), p_3^3 \lambda_3(X^q), p_4^3 \lambda_4(X^q)) = 0.8733$$

Other points from the Pareto region were obtained in a similar way $X_t^o = \{\lambda_t^o, X_t^o\} \in S^o$.



2.k1. Analysis of the Results of Solving a Vector Problem of Mathematical Programming - a Model of an Engineering System with a Given Priority of the First Criterion.

The calculated size of criterion $f_q(X_t^o) = 381.7, q = 1 \in K$ is usually not equal to the set $f_q = 380$. The error of the choice of $\Delta f_q = |f_q(X_t^o) - f_q| = |381.5 - 380| = 1.5$ is defined by an error of linear approximation, $\Delta f_{q\%} = 0.5\%$.

If error $\Delta f_q = |f_q(X_t^o) - f_q| = |381.5 - 380| = 1.7$, measured in physical units or as a percentage $\Delta f_{q\%} = \frac{\Delta f_q}{f_q} * 100 = 0.5\%$, is more than set $\Delta f, \Delta f_q > \Delta f$; we pass to a step 2, if $\Delta f_q \leq \Delta f$, calculations come to **the end**.

In the course of modelling parametrical restrictions (121) and function (120) can be changed, i.e. some set of optimum decisions is received. Choose a final version which in our example included from this set of optimum decisions: parameters of the technical sistem:

$X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}$;

the parameters of the technicum system at a given priority criterion $q=1$:

$X^q = \{x_1 = 46.3908, x_2 = 47.4561, x_3 = 5.9531, x_4 = 2.2000\}$.

3.k1. Geometric Interpretation of the Results of Solving the Vector Problem of Choosing Optimal Parameters by the First Priority Criterion in Relative Units.

For a geometric interpretation of the results of solving the vector problem of choosing the optimal parameters for solving the vector problem according to the first priority criterion in relative units, we will form Fig. 15.k1.

Similarly, Fig. 14, we form relative estimates of the four criteria at the optimum point X_k^* $\lambda_k(X_k^*), k = \overline{1, K}$ (black) and $\lambda_k^\Delta(X_k^*) = 1, k = \overline{1, K}$ (red colour) and present them in Fig. 15.k1.

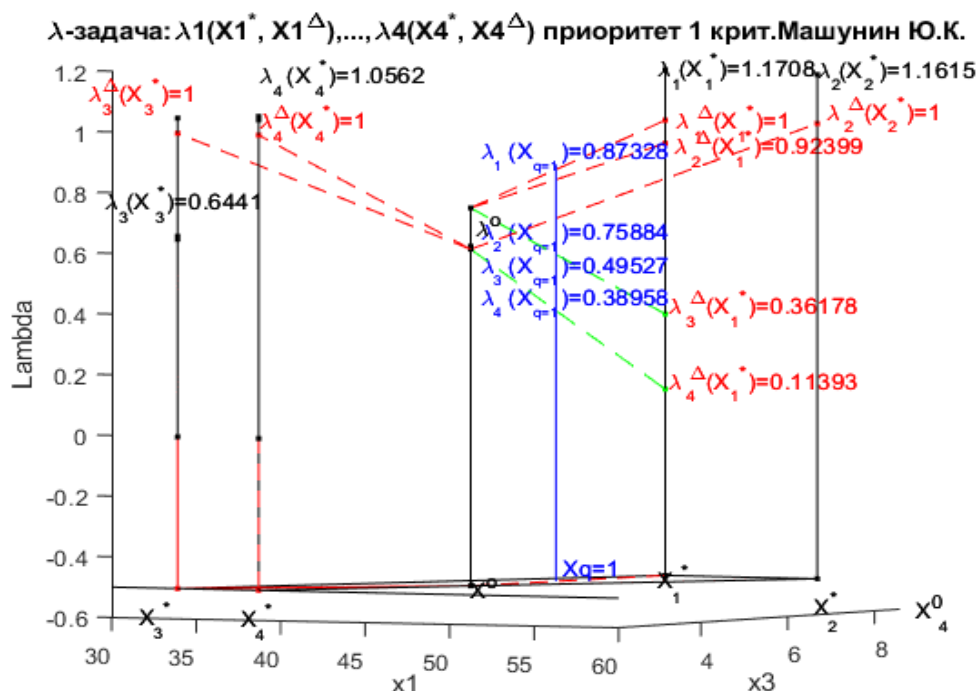


Figure 15.k1. λ -problem in the coordinate system x_1, x_3 and λ . Results of the solution: $\lambda_k^\Delta(X_k^*) = 1, k = \overline{1, K}$; relative estimates with the priority of the first criterion at the optimum point $X_{q=1}$: $\lambda_k(X_{q=1}), k = \overline{1, K}$

At the third step, the solution of the VPMP with equivalent criteria at the optimum points $X_1^*, X_2^*, X_3^*, X_4^*$ the values of all relative estimates are obtained:



$$L(X^*) = \begin{Bmatrix} L(X_1^*) \\ L(X_2^*) \\ L(X_3^*) \\ L(X_4^*) \end{Bmatrix} = \begin{Bmatrix} 1.00 & 0.9264 & 0.3618 & 0.1139 \\ 0.9364 & 1.00 & 0.1991 & 0.0326 \\ 0.0000 & 0.0000 & 1.00 & 0.1013 \\ 0.3931 & 0.2364 & 0.9364 & 1.00 \end{Bmatrix}.$$

Fig. 15.k1 $\lambda_1(X_1^*) = 1.0000$ is denoted as $\lambda_2^A(X_2^*)=1$.

Linear function connecting points $\lambda(X^o) = 0.7372$ and $\lambda_1^A(X_1^*)=1$ in relative units characterizes the function $f_1(X)$ in relative units in the four-dimensional dimension of the parameters $x_1, \dots, x_4$. This linear function represents a geometric interpolation of the functions $f_1(X)$ in relative units from N -dimensional (4-dimensional) to two-dimensional coordinate system. All functions (criteria) are studied in the same way.

In general, Fig. 15.k1 at X_1^* Geometric (linear) interpolation of all functions (criteria) is shown:

$f_1(X)$ and the corresponding relative assessment $\lambda_1^A(X_1^*) = 1.0$;

$f_2(X)$ and the corresponding relative assessment $\lambda_2^A(X_1^*) = 0.92451$;

$f_3(X)$ and the corresponding relative assessment $\lambda_3^A(X_1^*) = 0.35608$;

$f_4(X)$ and the corresponding relative assessment $\lambda_4^A(X_1^*) = 0.11216$.

Relative Estimates with the Priority of the First Criterion at the Optimum Point $X_{q=1}$

$$\lambda_k(X_{q=1}) = \frac{f_k(X_{q=1}) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}; \quad \lambda_k(X_{q=1}) = \{\lambda_1(X_{q=1}) = 0.87913, \lambda_2(X_{q=1}) = 0.76544, \lambda_3(X_{q=1}) = 0.48451, \lambda_4(X_{q=1}) = 0.3788\};$$

The results of the solution show that the mathematical results completely coincide with the geometric ones.

4.k1. Geometric interpretation of the results of the VPMP solution (functions: $f_1(X), \dots, f_4(X)$) with the priority of the first criterion – the model of the technical system when designing in a three-dimensional coordinate system in physical units.

The initial information for the geometric interpretation of the results of solving a vector problem (VPMP) with the priority of the first criterion is: parameters of the optimum point with equivalent criteria:

$$X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}$$

calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (see Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing. Let us examine and present the parameters sequentially for each characteristic of the technical systems (criterion): $f_1(X), f_2(X), f_3(X), f_4(X)$ in physical units.

1. Geometric interpretation of the results of the VPMP solution with the priority of the first criterion – the first characteristic $f_1(X)$ when designing in physical units.

In Figure 16.k1 investigated the optimum points $X_1^*, X_{q=1}$, with the corresponding relative estimates $\lambda_1(X_1^*) \lambda_1(X_{q=1})$. Linear functions in coordinates were also investigated:

$\lambda^o - \lambda_1(X_{q=1})$ and $\lambda^o - \lambda_1^A(X_1^*)$ in relative units, which characterize the function $f_1(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

The first characteristic of the technical systems $f_1(X)$ is formed in 4:

$$\max f_1(X) \equiv 323.84 - 2.25x_1 - 3.49x_2 + 10.72x_3 + 13.124x_4 + 0.0968x_1x_2 - 0.062x_1x_3 - 0.169x_1x_4 + 0.0743x_2x_3 - 0.1x_2x_4 - 0.0036x_3x_4 + 0.0143x_1^2 + 0.0118x_2^2 - 0.2434x_3^2 - 0.5026x_4^2.$$

Let us present the geometric interpretation of the function $f_1(X)$ in physical units with variable coordinates $\{x_1, x_3\}$ and with constant coordinates $\{x_2 = 49.99, x_4 = 2.2\}$ in Fig. 16.k1.



When calculated **by four variables**, $x_1, \dots, x_4$ coordinates of the point and function of the first criterion for four variables by **maximum**: $X_1^* = \{x_1 = 47.6, x_2 = 44.1968, x_3 = 8.0, x_4 = 2.2\}$, $f_1^* = f_1(X_1^*) = -392.9$, when calculated by four variables.

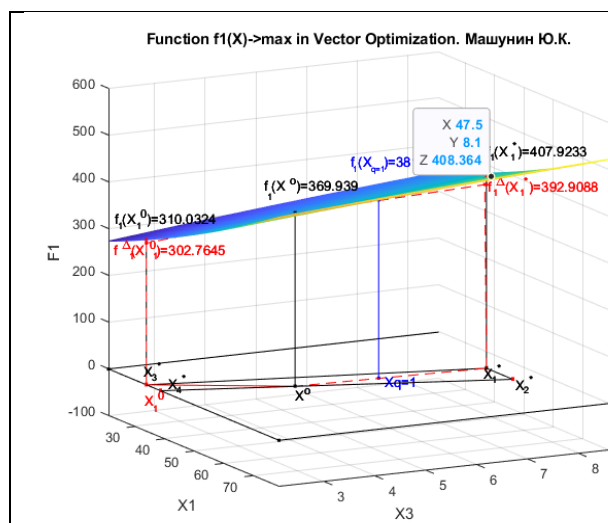


Figure 16.k1. Function $f_1(X)$ with the priority of **the first criterion** in the two-dimensional coordinate system $x_1 x_3$ and a geometric interpretation of the function $f_1(X_{1,q=1})$ in the coordinate system $x_1 x_2 x_3 x_4$

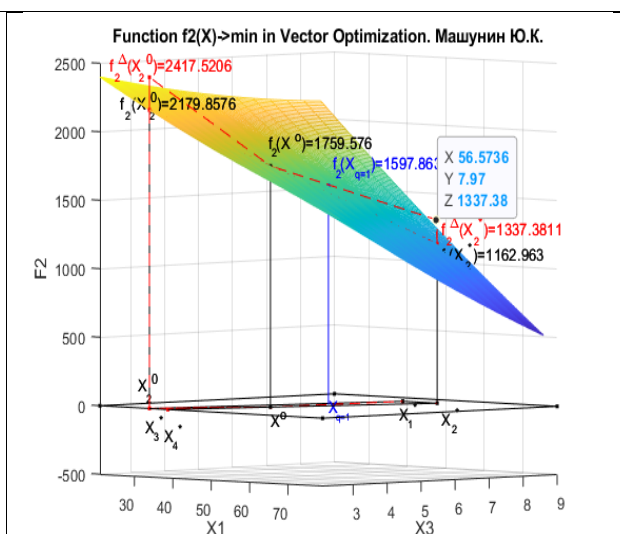


Figure 17.k1. The function $f_2(X)$ with the priority of **the first criterion** in the two-dimensional coordinate system $x_1 x_3$, geometric interpretation of the function $f_2(X_{1,q=1})$ are coordinates $x_1 x_2 x_3 x_4$ in physical units

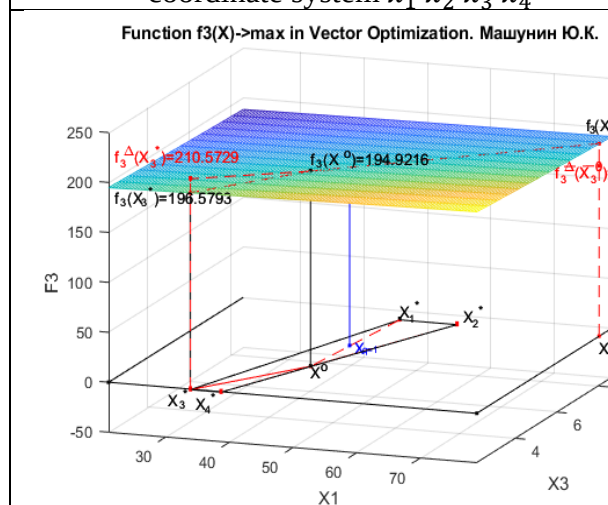


Figure 18.k1. Function $f_3(X)$ with the priority of **the first criterion** in the two-dimensional coordinate system $x_1 x_3$ and a geometric interpretation of the function $f_3(X_{1,q=1})$ in the coordinate system $x_1 x_2 x_3 x_4$

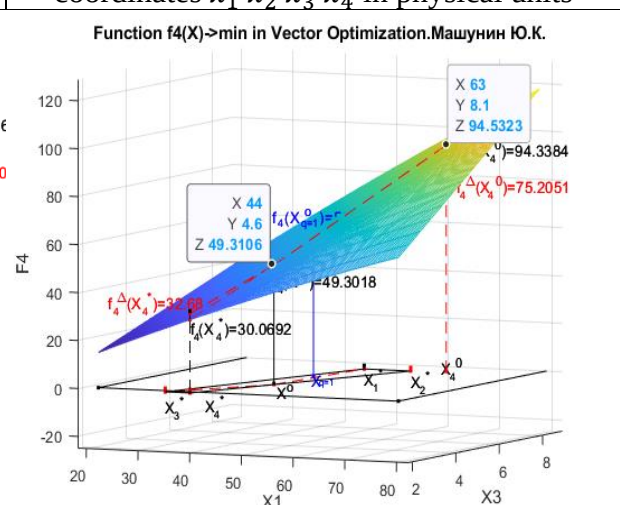


Figure 19.k1. Function $f_4(X)$ with the priority of **the first criterion** in the two-dimensional coordinate system $x_1 x_3$ and a geometric interpretation of the function $f_4(X)$ in the coordinate system $x_1 x_2 x_3 x_4$

In the figure 16.k1, it is denoted as $f_1^{\Delta}(X_1^*) = -392.9$;

$X_1^* = \{x_1 = 46.5676, x_3 = 8.0\}$ (in Fig. 16.k1 denoted as X_1^*). In a two-dimensional coordinate system y_1, y_3 .



$X_{10max} = \{x_1 = 47.6032, x_3 = 8.0\}$ (Fig. 16.k1) is denoted as X_1^* When calculated with two variables x_1, x_3 and two constants x_2, x_4 .

The value of the objective function is equal to $f_1^* = f_1(X_1^*) = 407.92$. (Black colour). In the MANLAB system, it is designated as $f_1(X_1^*) = 407.92$.

The coordinates of the point and the function of the *first criterion to the minimum*:

At the point $X_1^0 = \{x_1 = 31.9, x_2 = 59.0, x_3 = 2.1, x_4 = 7.0\}$ value of the objective function $f_1^0 = f_1(X_1^0) = 302.76$. Value of the objective function in a four-dimensional coordinate system $f_1^\Delta(X_1^0) = 302.76$. In the two-dimensional coordinate system x_1, x_3 the value of the objective function is equal to $FX_{10min} = 310.0$. (Black color). Point coordinates $X_{10min} = \{x_1 = 33.90, x_2 = 49.9964, x_3 = 2.10, x_4 = 2.1\}$, value of the objective function $FX_{10min} = 310.0$.

Coordinates of the point and function of the first criterion *with equivalent criteria*:

The coordinates of the point are $X^0 = \{x_1 = 45.445, x_3 = 4.3578\}$. Value of the objective function $f_1(X^0) = 369.9$. $\lambda^0 = \lambda_2(X^0) = \lambda_3(X^0) = 0.6091$.

Relative evaluation with priority of the first criterion at the optimum point $X_{q=1}$: $\lambda_1(X_{q=1}) = \frac{f_1(X_{q=1}) - f_1^0}{f_1^* - f_1^0} = 0.87586$.

In physical units, the value of the first criterion with priority at the optimum point is $X_{q=1}$ is equal to $f_1(X_{q=1}) = 381.5$ is close to the given $f_1(X_{q=1}) = 380$ (shown in purple).

A linear function connecting the points $f_1(X^0)$ and $f_1^\Delta(X_1^*)$ in physical units, it characterizes the function $f_1(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

And in general, the segments are $f_1^\Delta(X_1^*) - f_1(X^0) - f_1^\Delta(X_1^0)$ represent the geometric interpretation of the function $f_1(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

2. Geometric interpretation of the results of the solution of the VPMP - the second characteristic $f_2(X_{2,q=1})$ with the priority of the first criterion - the structure of the material in the design in physical units.

Fig. 17.k1 investigated: optimum points $X_2^*, X_{2,q=1}$, with corresponding relative estimates $\lambda_2(X_2^*)$ $\lambda_2(X_{2,q=1})$, as well as linear functions $\lambda^0 \lambda_2(X_{2,q=1}) - \lambda_2^\Delta(X_2^*)$ in relative units, which characterize the function $f_2(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

The second characteristic of the structure of the technical system $f_2(X)$ (132) is formed in 4.1.4:

$\min f_2(X) \equiv 795.72 + 23.89x_1 + 30.866x_2 - 25.8586x_3 - 45.0026x_4 - 0.7683x_1x_2 + 0.4703x_1x_3 + 0.7472x_1x_4 - 0.1283x_2x_3 + 0.3266x_2x_4 - 0.0112x_3x_4 + 0.0398x_1^2 + 0.0365x_2^2 + 3.2x_3^2 + 2.6457x_4^2$.

Let us present a geometric interpretation of the function $f_2(X)$ in physical units with variable coordinates $\{x_1, x_3\}$ and with constant coordinates $\{x_2 = 49.54, x_4 = 2.2\}$, taken from the optimal point X^0 (150) in Fig. 17.k1.

Coordinates of the point and the function of the second criterion on *the optimum (minimum)*:

$f_2^* = f_2(X_2^*) = 1337.4$. Parameters: $X_2^* = \{x_1 = 56.60, x_2 = 35.1964, x_3 = 8, x_4 = 2.2\}$ when calculated by four variables. In the figure, it is labelled as $f_2^\Delta(X_2^*) = 1337.4$.

In the MATLAB system, it is designated as $f_2^\Delta(X_2^*) = Z = 1337.38$.

$X_2^* = \{x_1 = 55.6, x_3 = 8.0\}$ (in Fig. 17.k1 denoted as X_2^*).

In a two-dimensional coordinate system x_1, x_3 . the value of the objective function is equal to $f_2^* = f_2(X_2^*) = 1163.06$. (Black colour).

The coordinates of *the minimum* point (Worst Case - Maximum) at the point

$X_2^0 = \{x_1 = 31.9, x_3 = 2.1\}$ value of the objective function $F_2^0 = f_2(X_2^0) = 2179.9$.

Value of the objective function in a four-dimensional coordinate system $f_2^\Delta(X_2^0) = 2417.5$.



The coordinates of the point and the function of **the second criterion with equivalent criteria**: The coordinates of the point are $X^o = \{x_1 = 43.9, x_3 = 4.348\}$. The value of the objective function is $f_2(X^o) = 1759.6$.

Relative evaluation with priority of the first criterion at the optimum point $X_{q=1}$ of the second criterion: $\lambda_2(X_{q=1}) = \frac{f_2(X_{q=1}) - f_2^0}{f_2^* - f_2^0} = 0.76574$.

In physical units, the value of the first criterion with priority at the optimum point is $X_{q=1}$ is equal to $f_2(X_{q=1}) = 1597.9$. (Shown in purple).

A linear function connecting the points $f_2(X^o)$ and $f_2^\Delta(X_2^*)$ in physical units, it characterizes the function $f_2(X)$ in the four-dimensional dimension of the parameters

$x_1, \dots, x_4$. And in general, the segments are $f_2^\Delta(X_2^*) - f_2(X^o) - f_2^\Delta(X_2^0)$ represent the geometric interpolation of the function $f_2(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

3. Geometric interpretation of the results of the solution of the VPMP – the third characteristic $f_3(X_{3,q=1})$ with the priority of **the first criterion** - the structure of the material in the design in physical units.

Figure 18.k1 investigated the optimum points X_3^* , $X_{3,q=1}$, with the corresponding relative estimates $\lambda_3(X_3^*)$ $\lambda_3(X_{3,q=1})$. Linear functions in coordinates were also investigated:

$\lambda^o \lambda_3(X_{3,q=1}) - \lambda_3^\Delta(X_3^*) \lambda^o$ in relative units, which In characterize the function $f_3(X_{3,q=1})$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

The third characteristic of the technical system $f_3(X)$ is formed in 4:

$$\max f_3(X) \equiv 110.22 + 0.7918x_1 + 1.73x_2 - 0.3713x_3 - 2.20x_4 - 0.0132x_1x_2 - 0.008x_1x_3 + 0.0193x_1x_4 - 0.0172x_2x_3 + 0.0161x_2x_4 - 0.0006x_3x_4 - 0.0004x_1^2 - 0.0002x_2^2 + 0.0335x_3^2 + 0.124x_4^2\}.$$

Let us present the geometric interpretation of the function $f_3(X)$ in physical units with variable coordinates $\{x_1, x_3\}$ and with constant coordinates $\{x_2 = 49.54, x_4 = 2.2\}$ in Fig. 18.k1. The coordinates of the point and the function of **the third criterion to the maximum**:

$$X_3^* = \{x_1 = 33.90, x_2 = 59.00, x_3 = 2.1, x_4 = 7.0\}, f_3^* = f_3(X_3^*) = -210.57,$$

when calculated by four variables. In the figure 18.k1, it is denoted as $f_3^\Delta(X_3^*) = -210.57$

$$X_3^* = \{x_1 = 33.9, x_3 = 2.1\} \text{ (in Fig. 18.k1 denoted as } X_3^* \text{)}.$$

In a two-dimensional coordinate system x_1, x_3 . The value of the objective function is equal to $f_3^* = f_3(X_3^*) = 196.83$. (Black colour). Value of the objective function $f_3^* = 196.83$.

The coordinates of the point and the function of **the third criterion to the minimum**:

At the point $X_3^0 = \{x_1 = 78.16, x_3 = 8.0\}$ value of the objective function

$f_3^0 = f_3(X_3^0) = 192.69$. Value of the objective function in a four-dimensional coordinate system $f_3^\Delta(X_3^0) = 170.53$. Coordinates of the point and function of the third criterion **with equivalent criteria**: The coordinates of the point are $X^o = \{x_1 = 43.9, x_3 = 4.348\}$. Value of the objective function $f_3(X^o) = 194.27$. Relative evaluation with priority of the first criterion at the optimum point

$$X_{q=1} \text{ third criterion: } \lambda_3(X_{q=1}) = \frac{f_3(X_{3,q=1}) - f_3^0}{f_3^* - f_3^0} = 0.48683.$$

In physical units, the value of the third criterion with priority at the optimum point $X_{3,q=1}$ is close to the given $f_3(X_{q=1}) = 194.9$. (shown in purple).

A linear function connecting the points $f_3(X^o)$ and $f_3^\Delta(X_3^*)$ in physical units, it characterizes the function $f_3(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

And in general, the segments are $f_3^\Delta(X_3^*) - f_3(X^o) - f_3^\Delta(X_3^0)$ represent the geometric interpretation of the function $f_3(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.



4. Geometric interpretation of the results of the solution of the VPMP – the fourth characteristic $f_4(X_{4,q=1})$ with the priority of the first criterion of the material structure in the design in physical units.

Figure 19.k1 investigated the optimum points X_4^* , $X_{4,q=1}$, with the corresponding relative estimates $\lambda_4(X_4^*)$, $\lambda_4(X_{4,q=1})$. Linear functions in coordinates were also investigated:

$\lambda^0 - \lambda_4(X_{4,q=1})$ and $\lambda^0 - \lambda_4^{\Delta}(X_4^*)$ in relative units, which characterize the function $f_4(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

The **fourth** characteristic of the structure of the material $f_4(X)$:

$$\max f_4(X) \approx 21.004 - 0.0097x_1 - 0.841x_2 - 0.4326x_3 + 1.1723x_4 + 0.166x_1x_2 + 0.085x_1x_3 - 0.0001x_1x_4 + 0.0523x_2x_3 + 0.0002x_2x_4 + 0.0006x_3x_4 - 0.0022x_1^2 + 0.0035x_2^2 + 0.006x_3^2 - 0.0311x_4^2\}.$$

Let us present the geometric interpretation of the function $f_4(X_{4,q=1})$ in physical units with variable coordinates $\{x_1, x_3\}$ and with constant coordinates $\{x_2 = 49.54, x_4 = 2.2\}$ in Fig. 19.k1.

The coordinates of the point and the function of the *fourth criterion to the maximum*:

$$X_4^* = \{x_1 = 38.70, x_2 = 59.00, x_3 = 2.1, x_4 = 2.2\}, f_4^* = f_4(X_4^*) = 32.68$$

when calculated by four variables. In the figure 19.k1, it is denoted as $f_4^{\Delta}(X_4^*) = 32.68$;

$X_4^* = \{x_1 = 38.70, x_3 = 2.1\}$. In a two-dimensional coordinate system x_1, x_3 the value of the objective function is equal to $f_4^* = f_4(X_4^*) = 30.1$. (Black colour). The value of the objective function is equal to $F_4^* = 30.1$.

The coordinates of the point and the function of the *fourth criterion to the minimum*:

At the point $X_4^0 = \{x_1 = 63.46, x_3 = 8\}$. Value of the objective function in a four-dimensional coordinate system $f_4^0 = f_4^{\Delta}(X_4^0) = -75.205$.

Coordinates of the point and function of the fourth criterion *with equivalent criteria*:

The coordinates of the point are $X^0 = \{x_1 = 45.4458, x_3 = 4.3578\}$. Value of the objective function $f_4(X^0) = 49.3$.

Relative evaluation with priority of the first criterion at the optimum point $X_{q=1}$:

$\lambda_4(X_{q=1}) = \frac{f_4(X_{q=1}) - f_4^0}{f_4^* - f_4^0} = 0.38154$. In physical units, the value of the fourth criterion with priority at the optimum point is $X_{q=1}$ is equal to $f_4(X_{q=1}) = 58.6$ (shown in purple).

A linear function connecting the points $f_4(X^0)$ and $f_4^{\Delta}(X_4^*)$ in physical units, it characterizes the function $f_4(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

And in general, the segments are $f_4^{\Delta}(X_4^*) - f_4(X^0) - f_4^{\Delta}(X_4^0)$ represent the geometric interpretation of the function $f_4(X)$ in the four-dimensional dimension of the parameters $x_1, \dots, x_4$.

Conclusion on the section: criterion k1.

The section considers and solves the problem (fragment) of the development and adoption of a management decision under conditions of uncertainty in a complex engineering system (technical system). The analysis of the results of the solution of the VPMP at the given priority of the **first criterion** is carried out, a geometric interpretation of the results of the solution when designing four characteristics (criteria) in a three-dimensional coordinate system is presented, firstly, in relative units, and secondly, in physical units.

3.3.5. Block k2. Selection of optimal parameters of the engineering system (technical system) according to the second priority criterion, geometric interpretation of the results of the VPMP solution, in relative and physical units

The initial information for the geometric interpretation of the results of solving a vector problem with the priority of the second criterion is: parameters of the optimum point with equivalent criteria: $X^0 = \{X^0, \lambda^0\} = \{X^0 = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^0 =$



0.6091} calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (see Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing.

The analysis of the results of the VPMP solution at the given priority of the second criterion is carried out and a geometric interpretation of the solution results when designing in a three-dimensional coordinate system **in relative units** is presented.

A geometric interpretation of the results of the VPMP solution with the priority of the second criterion when designing in a three-dimensional coordinate system **in physical units** is presented.

1.k2. Solution of VPMP - a model of a complex engineering system (technical system) with a given priority of the second criterion in relative units in multidimensional mathematics.

The person making decisions, as a rule, is the designer of the technical system.

Step 1. The solution of a VPMP with equivalent criteria.

Results of the decision are presented in Stage 4.

Step 2. Choice of priority criterion of $q \in K$.

From the pair $\lambda^0 = \lambda_2(X^0) = \lambda_3(X^0) = 0.6091$ contradictory criteria, the criterion that the decision-maker would like to improve is selected. The criterion is called the "priority criterion", $q = 2 \in K$. It is studied in interaction with the criterion $q = 1 \in K$.

Priority criterion selection: The display provides general information for decision-making: Criteria at the optimum point:

$$X^0 = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^0 = 0.6091\};$$

$$f_k(X^0) = \{f_1(X^0) = 369.9, f_2(X^0) = \mathbf{1759.6}, f_3(X^0) = 194.9, f_4(X^0) = 49.3\}. \quad (170)$$

Relative estimates in X^0 :

$$LX^0 = \{\lambda_1(X^0) = 0.7452 \quad \lambda_2(X^0) = \mathbf{0.6091} \quad \lambda_3(X^0) = 0.6091 \quad \lambda_4(X^0) = 0.6091\} \quad (171)$$

We examine these two criteria from the set of $K = 4$ criteria shown in Fig. 15.k2. A message is displayed on the display: $q = \text{input('Enter priority criterion (number) } q = \text{'})}$ – Enter criterion **$q=2$** .

Step 3. Numerical limits for changing the priority value of criterion q are formed:

$$f_2(X^0) = 1759.68 \leq f_2(X) \leq 1337.38 = f_2(X_2^*), q \in K. \quad (172)$$

Similarly, in relative units, the criterion $q=2$ varies within the following limits:

$$\lambda_2(X^0) = 0.6091 \leq \lambda_2(X) \leq 1 = \lambda_2(X_2^*), q = 2 \in K. \text{ These data are analyzed.}$$

Step 4. Choice of size of priority criterion of $q \in K$. (Decision-making).

On the message: "Enter the size of priority criterion f_q " - we enter, the size of the characteristic defining structure of material: **$f_q = 1500$** .

$$f_2(X^0) = -1759.6 \leq f_2(X) = \mathbf{1500} \leq f_2(X_2^*) = -1337.38. \text{ Next, Step 5, ..., Step 8.}$$

Geometric interpretation of the results of solving the vector problem of choosing optimal parameters according to the second priority criterion in relative units. Similarly, Fig. In Figure 15.k1, we form the relative estimates of the four criteria at the optimum point X_k^* $\lambda_k(X_k^*)$, $k = \overline{1, K}$ (black) and $\lambda_k^A(X_k^*) = 1$, $k = \overline{1, K}$ (red color). Let's imagine their Fig. 15.k2.

Calculation of the main indexes of a point of X^q .

For the received X^q point, we will calculate:

all criteria in physical units $f_k(X^q) = \{f_k(X^q), k = \overline{1, K}\}$,

$$f(X^q) = \{f_1(X^q) = 381.3, f_2(X^q) = \mathbf{1465.7}, f_3(X^q) = 182.1, f_4(X^q) = 64.5\};$$

all relative estimates of criteria: $\lambda^q = \{\lambda_k^q, k = \overline{1, K}\}$, $\lambda_k(X^q) = \frac{f_k(X^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}$,

$$\lambda_k(X^{q=2}) = \{\lambda_1(X^q) = 0.9189, \quad \lambda_2(X^q) = 0.8813, \lambda_3(X^q) = 0.3385,$$

$$\lambda_4(X^q) = 0.2461\};$$



λ -задача: $\lambda_1(X_1^*, X_1^\Delta), \dots, \lambda_4(X_4^*, X_4^\Delta)$ приоритет 1 крит. Машунин Ю.К.

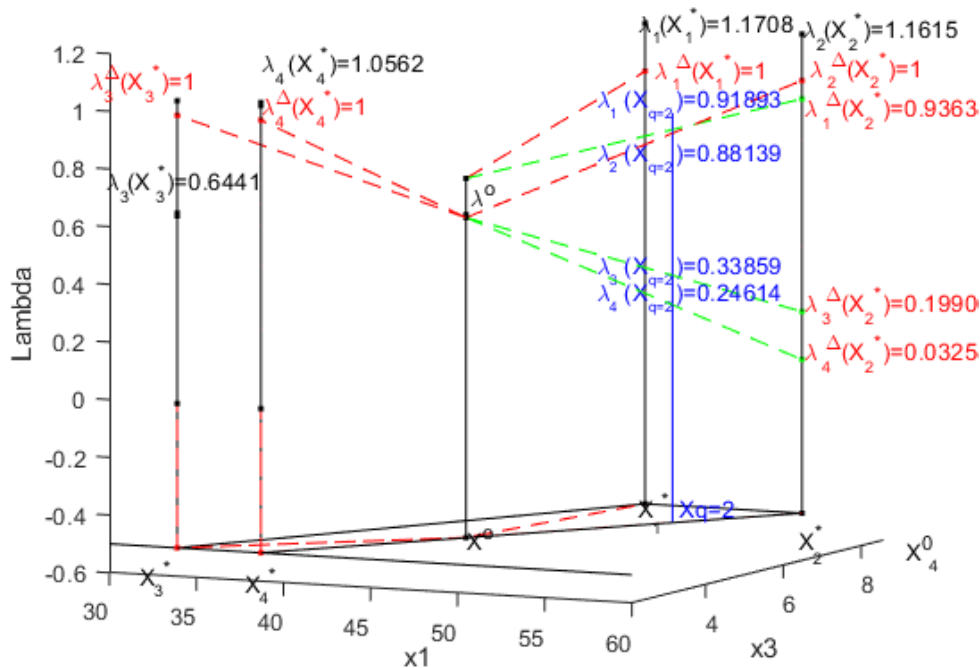


Figure 15.k2. λ -problem in the coordinate system x_1, x_3 and λ . Results of the solution: $\lambda_k^\Delta(X_k^*) = 1, k = \overline{1, K}$; relative estimates with the priority of the second criterion at the optimum point $X_{q=2}$:

$$\lambda_k(X_{q=2}), k = \overline{1, K}$$

2.k2. Analysis of results. The calculated size of criterion

$f_q(X_t^o) = 1465.7, q = 2 \in K$ is usually not equal to the set $f_q = 1500$. The error of the choice of $\Delta f_q = |f_q(X_t^o) - f_q| = |1465.7 - 1500| = 34.3$ is defined by an error of linear approximation, $\Delta f_{q\%} = 0.5\%$.

If error $\Delta f_q = |f_q(X_t^o) - f_q| = |1465.7 - 1500| = 34.3$, measured in physical units or as a percentage $\Delta f_{q\%} = \frac{\Delta f_q}{f_q} * 100 = 0.5\%$, is more than set $\Delta f, \Delta f_q > \Delta f$; we pass to a step 2, if $\Delta f_q \leq \Delta f$, calculations come to **the end**.

k2. Geometric interpretation of the results of the VPMP solution with the priority of the second criterion – the model of the technical system when designing in a three-dimensional coordinate system in physical units.

The initial information for the geometric interpretation of the results of solving a vector problem (VPMP) with the priority of the first criterion is: parameters of the optimum point with equivalent criteria: $X^o = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}$:

calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (see Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing.



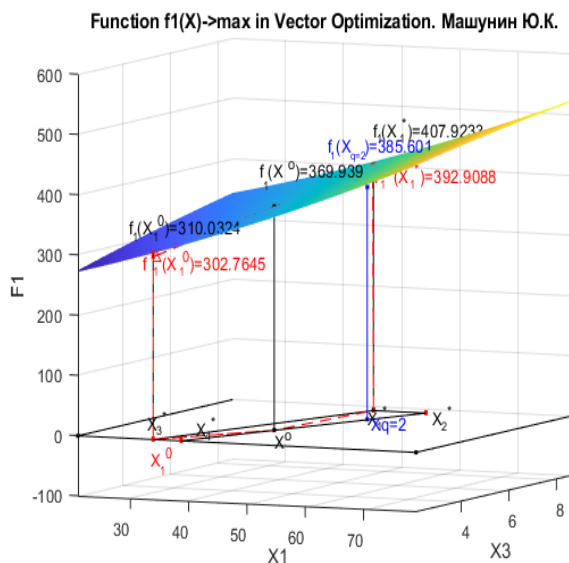


Figure 16.k2. Function $f_1(X)$ **with the priority of the second criterion** in the two-dimensional coordinate system $x_1 x_3$ and a geometric interpretation of the function $f_1(X_{2,q=2})$ in the coordinate system $x_1 x_2 x_3 x_4$

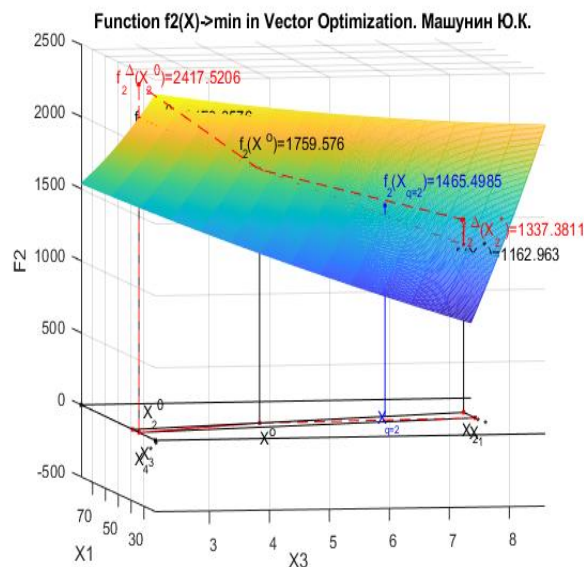


Figure 17.k2. The function $f_2(X)$ **with the priority of the second criterion** in the two-dimensional coordinate system $x_1 x_3$, geometric interpretation of the function $f_2(X_{2,q=2})$ are coordinates $x_1 x_2 x_3 x_4$

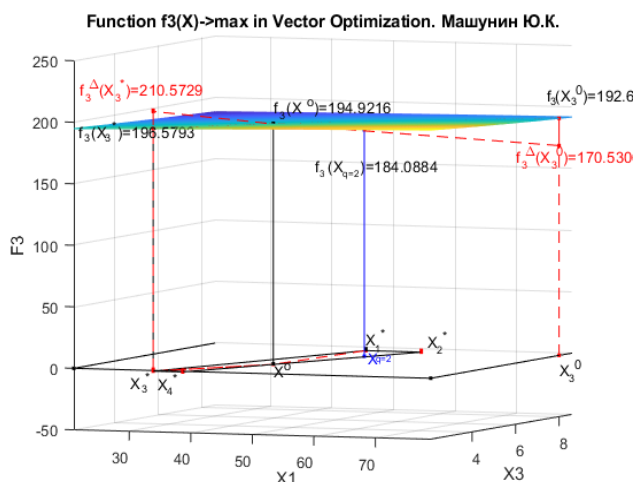


Figure 18.k2. Function $f_3(X)$ **with the priority of the second criterion** in the two-dimensional coordinate system $x_1 x_3$ and a geometric interpretation of the function $f_3(X_{2,q=2})$ in the coordinate system $x_1 x_2 x_3 x_4$

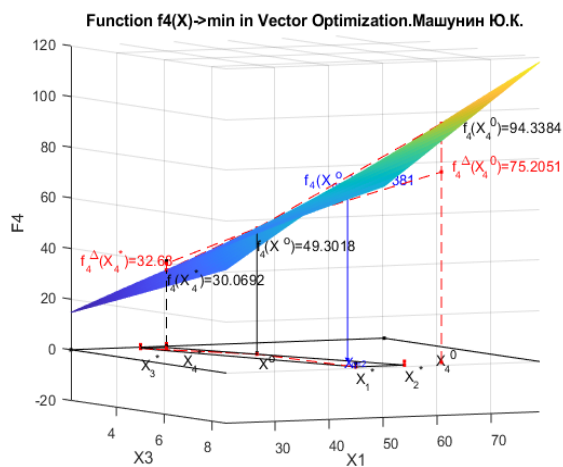


Figure 19.k2. Function $f_4(X)$ **with the priority of the second criterion** in the two-dimensional coordinate system $x_1 x_3$ and a geometric interpretation of the function $f_4(X_{2,q=2})$ in the coordinate system $x_1 x_2 x_3 x_4$

By analogy with the drawings of Fig. 16.k1., ..., Fig. 19.k1, we will present:



Fig. 16.k2 the first criterion $f_1(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_1(X)$ with the priority of the second criterion $f_{q=2}=1500$;

Fig. 17.k2 the second criterion $f_2(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_2(X)$ with the priority of the second criterion $f_{q=2}=1500$;

Fig. 18.k2 the third criterion $f_3(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_3(X)$ with the priority of the second criterion $f_{q=2}=1500$;

Fig. 19.k2 the found criterion $f_4(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_4(X)$ with the priority of the second criterion $f_{q=2}=1500$.

3.3.6. Block k3. Selection of optimal parameters of the engineering system (technical system) according to the third priority criterion, geometric interpretation of the results of the VPMP solution, in relative and physical units

The initial information for the geometric interpretation of the results of solving a vector problem with the priority of the third criterion is: parameters of the optimum point with equivalent criteria:

$$X^0 = \{X^0, \lambda^0\} = \{X^0 = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^0 = 0.6091\}.$$

The analysis of the results of the VPMP solution at the given priority of the **third** criterion is carried out and a geometric interpretation of the solution results when designing in a three-dimensional coordinate system **in relative units** is presented.

A geometric interpretation of the results of the VPMP solution with the priority of the third criterion when designing in a three-dimensional coordinate system **in physical units** is presented.

1.k3. Solution of VPMP - a model of a complex engineering system (technical system) with a given priority of the third criterion in relative units in multidimensional mathematics.

The person making decisions, as a rule, is the designer of the technical system.

Step 1. The solution of a VPMP with equivalent criteria.

Results of the decision are presented in Stage 4.

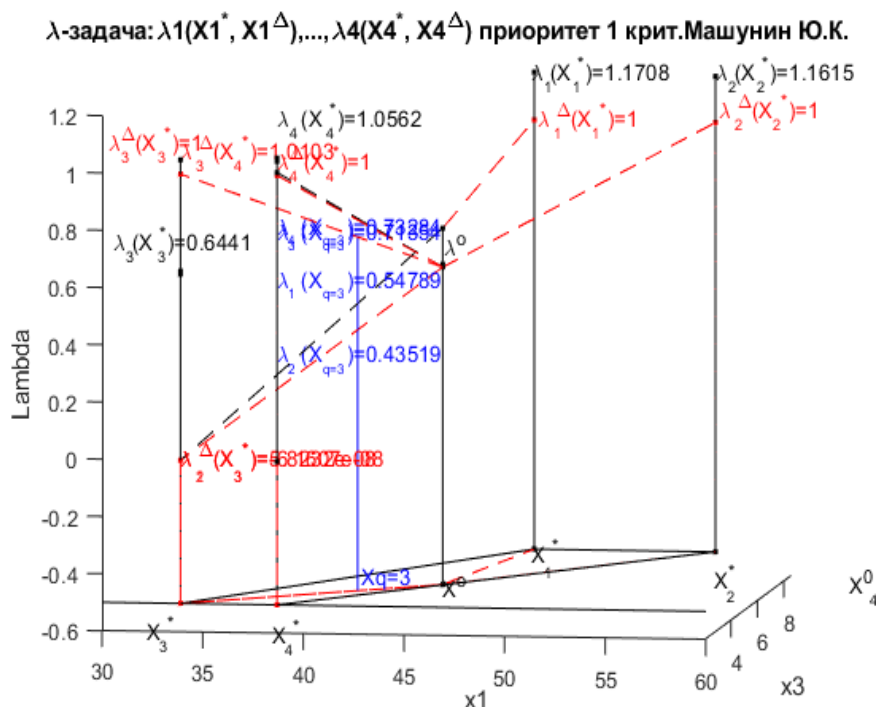


Figure 15.k3. λ -problem in the coordinate system x_1, x_3 and λ . Results of the solution: $\lambda_k^\Delta(X_k^*) = 1, k = \overline{1, K}$; relative estimates with the priority of the third criterion at the optimum point $X_{q=3}$:

$$\lambda_k(X_{q=3}), k = \overline{1, K}$$



Step 2. Choice of priority criterion of $q \in K$.

From the pair $\lambda^0 = \lambda_2(X^0) = \lambda_3(X^0) = 0.6091$ contradictory criteria, the criterion that the decision-maker would like to improve is selected. The criterion is called the "priority criterion", $q = 3 \in K$. It is studied in interaction with the criterion $q = 1 \in K$.

Priority criterion selection: The display provides general information for decision-making: Criteria at the optimum point:

$$X^0 = \{x_1 = 45.445, x_2 = 49.99,$$

$$x_3 = 4.3, x_4 = 2.2, \lambda^0 = 0.6091\};$$

$$f_k(X^0) = \{f_1(X^0) = 369.9, f_2(X^0) = 1759.6, f_3(X^0) = \mathbf{194.9}, f_4(X^0) = 49.3\}.$$

$$\text{Relative estimates in } X^0: \text{LXo} = \{0.7452 \quad 0.6091 \quad \lambda_3(X^0) = \mathbf{0.6091} \quad 0.6091\}.$$

We examine these two criteria from the set of $K = 4$ criteria shown in Fig. 15.k3.

A message is displayed on the display: $q = \text{input}(\text{'Enter priority criterion (number) } q = \text{'})$ – Enter criterion $q=3$.

Step 3. Numerical limits for changing the priority value of criterion q are formed:

$$f_3(X^0) = 194.9 \leq f_3(X) = \mathbf{200} \leq f_3(X_1^*) = -210.57.$$

Similarly, in relative units, the criterion $q=1$ varies within the following limits:

$$\lambda_3(X^0) = 0.6091 \leq \lambda_3(X) \leq 1 = \lambda_3(X_2^*), q = 3 \in K. \text{ These data are analyzed.}$$

Step 4. Choice of size of priority criterion of $q=4 \in K$. (Decision-making).

On the message: "Enter the size of priority criterion $f_q =$ " - we enter, the size of the characteristic defining structure of material: $f_{q=3} = \mathbf{200}$.

$$f_3(X^0) = 194.9 \leq f_3(X) = \mathbf{200} \leq f_3(X_1^*) = -210.57.$$

Next, **Step 5, ..., Step 8.**

Geometric interpretation of the results of solving the vector problem of choosing optimal parameters according to the third priority criterion in relative units.

Similarly, Fig. In Figure 15.k1, we form the relative estimates of the four criteria at the optimum point $X_k^* \lambda_k(X_k^*), k = \overline{1, K}$ (black) and $\lambda_k^\Delta(X_k^*) = 1, k = \overline{1, K}$ (red color).

Solution results: relative estimates of four criteria with the priority of the third criterion at the optimum point $X_{q=3} - \lambda_k(X_{q=3}), k = \overline{1, K}: \lambda_1(X_{q=3}) = 0.7398, \dots, \lambda_4(X_{q=3}) = 0.4351$. (Purple). Let's imagine their Fig. 15.k3.

4.k3. Geometric interpretation of the results of the VPMP solution with the priority of the third criterion – the model of the technical system when designing in a three-dimensional coordinate system in physical units.

The initial information for the geometric interpretation of the results of solving a vector problem (VPMP) with the priority of the third criterion is: parameters of the optimum point with equivalent criteria:

$$X^0 = \{X^0, \lambda^0\} = \{X^0 = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^0 = 0.6091\},$$

calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (see Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing.

By analogy with the drawings of Fig. 16.k1., ..., Fig. 19.k1, we will present:

Fig. 16.k3 the first criterion $f_1(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_1(X)$ with the priority of the third criterion $f_{q=3}=200$;

Fig. 17.k3 the second criterion $f_2(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_2(X)$ with the priority of the third criterion $f_{q=3}=200$;

Fig. 18.k3 the third criterion $f_3(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_3(X)$ with the priority of the third criterion $f_{q=3}=200$;



Fig. 19.k2 the found criterion $f_4(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_4(X)$ with the priority of the third criterion $f_{q=3}=200$.

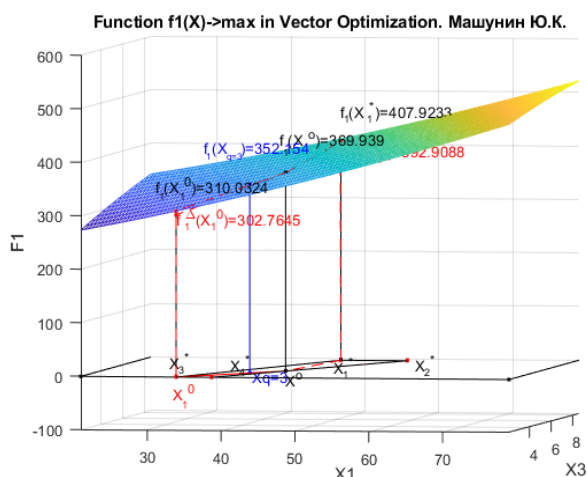


Figure 16.k3. Function $f_1(X)$ **with the priority of the third** criterion in the two-dimensional coordinate system $x_1 x_3$ and a geometric interpretation of the function $f_1(X_{3,q=3})$ in the coordinate system $x_1 x_2 x_3 x_4$

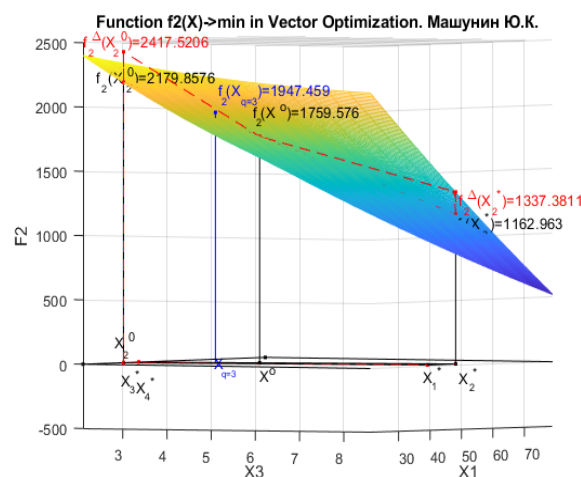


Figure 17.k3. The function $f_2(X)$ **with the priority of the third** criterion in the two-dimensional coordinate system $x_1 x_3$, geometric interpretation of the function $f_2(X_{3,q=3})$ are coordinates $x_1 x_2 x_3 x_4$

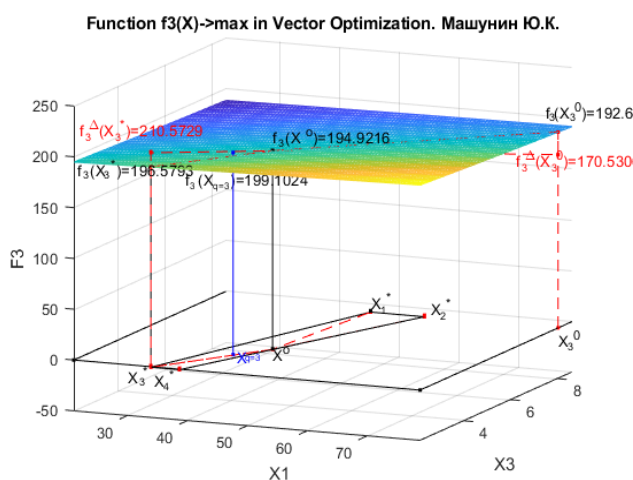


Figure 18.k3. Function $f_3(X)$ **with the priority of the third criterion** in the two-dimensional coordinate system $x_1 x_3$ and a geometric interpretation of the function $f_3(X_{3,q=3})$ in the coordinate system $x_1 x_2 x_3 x_4$

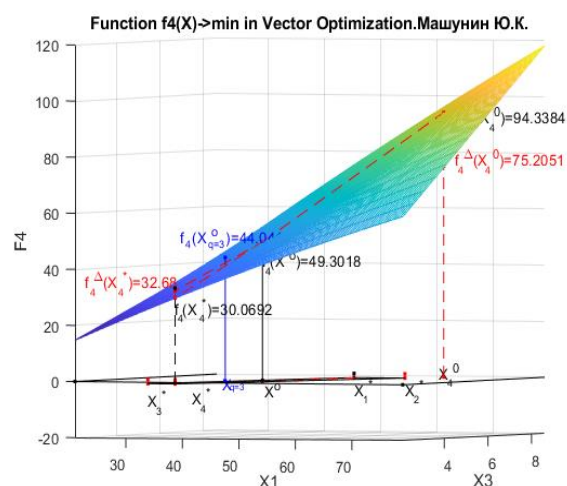


Figure 19.k3. Function $f_4(X)$ **with the priority of the third criterion** in the two-dimensional coordinate system $x_1 x_3$ and a geometric interpretation of the function $f_4(X_{3,q=3})$ in the coordinate system $x_1 x_2 x_3 x_4$

Conclusion on the section: criterion k3.



The section considers and solves the problem (fragment) of the development and adoption of a management decision under conditions of uncertainty in a complex engineering system (technical system). The analysis of the results of the solution of the VPMP at the given priority of **the third criterion** is carried out, a geometric interpretation of the results of the solution when designing four characteristics (criteria) in a three-dimensional coordinate system is presented, firstly, in relative units, and secondly, in physical units.

3.3.7. Block 4.k4. Selection of optimal parameters of the engineering system (technical system) according to the fourth priority criterion, geometric interpretation of the results of the VPMP solution, in relative and physical units.

The initial information for the geometric interpretation of the results of solving a vector problem with the priority of the fourth criterion is: parameters of the optimum point with equivalent criteria: $X^o = \{X^o, \lambda^o\} = \{X^o = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^o = 0.6091\}$ calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (see Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing.

The analysis of the results of the VPMP solution at the given priority of the **fourth** criterion is carried out and a geometric interpretation of the solution results when designing in a three-dimensional coordinate system **in relative units** is presented.

A geometric interpretation of the results of the VPMP solution with the priority of the fourth criterion when designing in a three-dimensional coordinate system **in physical units** is presented.

1.k4. Solution of VPMP - a model of a complex engineering system (technical system) with a given priority of the fourth criterion in relative units in multidimensional mathematics.

The person making decisions, as a rule, is the designer of the technical system.

Step 1. The solution of a VPMP with equivalent criteria.

Results of the decision are presented in Stage 4.

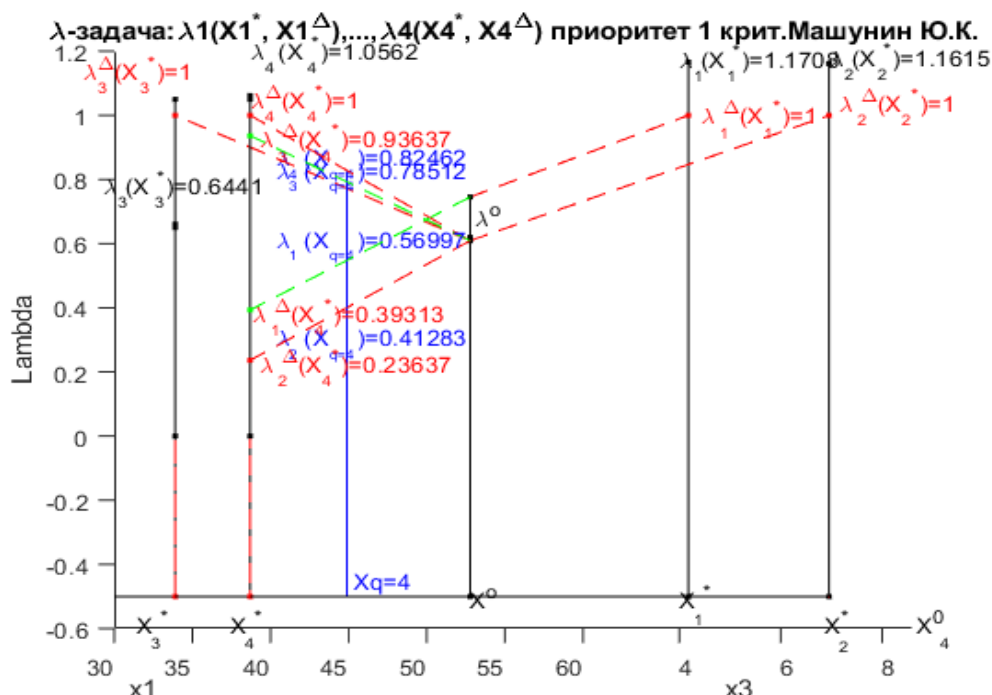


Figure 15.k4. λ -problem in the coordinate system x_1, x_3 and λ .

Results of the solution: $\lambda_k^{\Delta}(X_k^*) = 1, k = \overline{1, K}$; relative estimates with the priority of the fourth criterion at the optimum point $X_{q=4}$: $\lambda_k(X_{q=4}), k = \overline{1, K}$



Step 2. Choice of priority criterion of $q \in K$.

From the pair $\lambda^0 = \lambda_2(X^0) = \lambda_4(X^0) = 0.6091$ contradictory criteria, the criterion that the decision-maker would like to improve is selected. The criterion is called the "priority criterion", $q = 4 \in K$. It is studied in interaction with the criterion $q = 1 \in K$.

Priority criterion selection: The display provides general information for decision-making: Criteria at the optimum point: $X^0 = \{x_1 = 45.445, x_2 = 49.99,$

$$x_3 = 4.3, x_4 = 2.2, \lambda^0 = 0.6091\};$$

$$f_k(X^0) = \{f_1(X^0) = 369.9, \\ f_2(X^0) = 1759.6, f_3(X^0) = 194.9,$$

$$f_4(X^0) = 49.3\}.$$

Relative estimates in X^0 : $LX^0 = \{0.7452 \quad 0.6091 \quad 0.6091 \quad \lambda_3(X^0) = 0.6091\}$.

We examine these two criteria from the set of $K = 4$ criteria shown in Fig. 15.k4. A message is displayed on the display: $q = \text{input('Enter priority criterion (number) } q = \text{'})}$ – Enter criterion $q = 4$.

Step 3. Numerical limits for changing the priority value of criterion q are formed:

$$f_{q=4}(X^0) = 47.5045 \leq f_{q=4}(X) \leq 32.68 = f_q(X_q^*), q \in K.$$

Similarly, in relative units, the criterion $q=4$ varies within the following limits:

$$\lambda_4(X^0) = 0.6091 \leq \lambda_4(X) \leq 1 = \lambda_4(X_4^*), q = 4 \in K.$$

These data are analyzed.

Step 4. Choice of size of priority criterion of $q=4 \in K$. (Decision-making).

On the message: "Enter the size of priority criterion $f_q =$ " - we enter, the size of the characteristic defining structure of material: $f_{q=4} = 40$.

$$f_4(X^0) = -49.3 \geq f_4(X) = 40 \geq f_4(X_1^*) = -32.68.$$

Next, **Step 5, ..., Step 8.**

Geometric interpretation of the results of solving the vector problem of choosing optimal parameters according to the fourth priority criterion in relative units.

Similarly, Fig. In Figure 15.k4, we form the relative estimates of the four criteria at the optimum point X_k^* $\lambda_k(X_k^*)$, $k = \overline{1, K}$ (black) and $\lambda_k^A(X_k^*) = 1$, $k = \overline{1, K}$ (red color).

Solution results: relative estimates of four criteria with the priority of the third criterion at the optimum point $X_{q=4}^*$ $\lambda_k(X_{q=4}^*)$, $k = \overline{1, K}$: $\lambda_1(X_{q=4}^*) = 0.8246, \dots, \lambda_4(X_{q=4}^*) = 0.4128$. (Purple). Let's imagine their Fig. 15.k4.

4.k4. Geometric interpretation of the results of the VPMP solution with the priority of the fourth criterion $f_{q=4}=40$ – the model of the technical system when designing in a three-dimensional coordinate system in physical units.

The initial information for the geometric interpretation of the results of solving a vector problem (VPMP) with the priority of the fourth criterion is: parameters of the optimum point with equivalent criteria:

$$X^0 = \{X^0, \lambda^0\} = \{X^0 = \{x_1 = 45.4458, x_2 = 49.9964, x_3 = 4.3578, x_4 = 2.2, \lambda^0 = 0.6091\},$$

calculated at the fifth step of the algorithm in the two-dimensional coordinate system x_1, x_3 (Fig. 11) and presented in a three-dimensional coordinate system x_1, x_3 and λ in relative units in Figs. 12, 13, 14 when designing.

By analogy with the drawings of Fig. 16.k1., ..., Fig. 19.k1, we will present:

Fig. 16.k4 the first criterion $f_1(X)$ in physical units when designed in a three-dimensional coordinate system x_1, x_3 , $f_1(X)$ with the priority of the fourth criterion $f_{q=4}=40$;

Fig. 17.k4 the second criterion $f_2(X)$ in physical units when designed in a three-dimensional coordinate system x_1, x_3 , $f_2(X)$ with the priority of the fourth criterion $f_{q=4}=40$;



Fig. 18.k4 the third criterion $f_3(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_3(X)$ with the priority of the *fourth* criterion $f_{q=4}=40$;

Fig. 19.k4 the found criterion $f_4(X)$ in physical units when designed in a three-dimensional coordinate system $x_1, x_3, f_4(X)$ with the priority of the *fourth* criterion $f_{q=4}=40$.

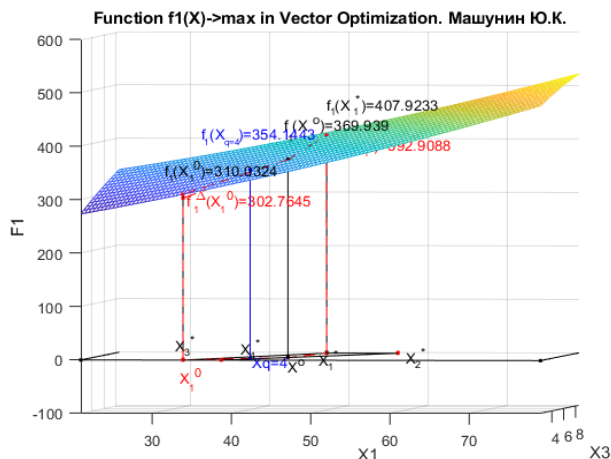


Figure 16.k4. Function $f_1(X)$ **with the priority of the fourth** criterion in the two-dimensional coordinate system x_1, x_3 and a geometric interpretation of the function $f_1(X_{4,q=4})$ in the coordinate system x_1, x_2, x_3, x_4

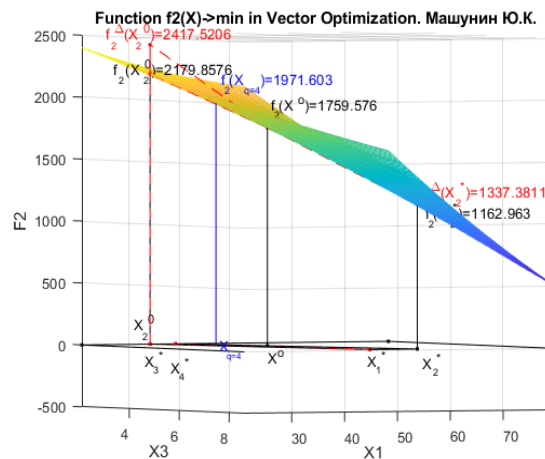


Figure 17.k4. The function $f_2(X)$ **with the priority of the fourth** criterion in the two-dimensional coordinate system x_1, x_3 , geometric interpretation of the function $f_2(X_{4,q=4})$ are coordinates x_1, x_2, x_3, x_4

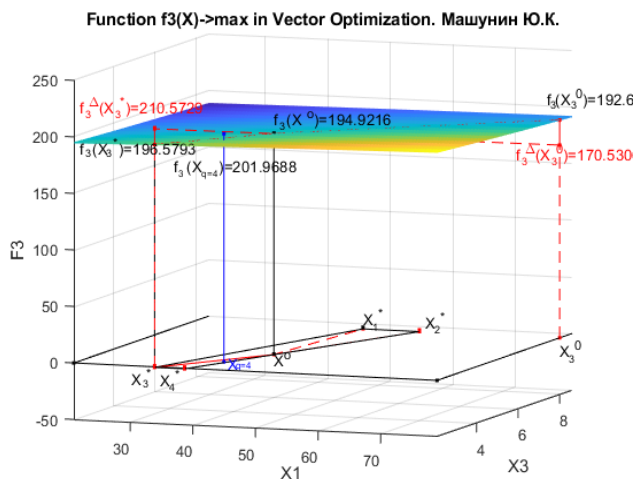


Figure 18.k4. Function $f_3(X)$ **with the priority of the fourth criterion** in the two-dimensional coordinate system x_1, x_3 and a geometric interpretation of the function $f_3(X_{4,q=4})$ in the coordinate system x_1, x_2, x_3, x_4

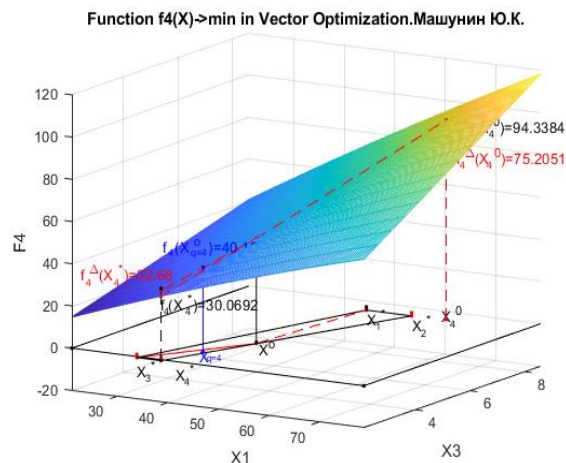


Figure 19.k4. Function $f_4(X)$ **with the priority of the fourth criterion** in the two-dimensional coordinate system x_1, x_3 and a geometric interpretation of the function $f_4(X_{4,q=4})$ in the coordinate system x_1, x_2, x_3, x_4

Conclusion on the section: criterion k4.

The section considers and solves the problem (fragment) of the development and adoption



of a management decision under conditions of uncertainty in a complex engineering system (technical system). The analysis of the results of the solution of the VPMP at the given priority of **the fourth criterion** is carried out, a geometric interpretation of the results of the solution when designing four characteristics (criteria) in a three-dimensional coordinate system is presented, firstly, in relative units, and secondly, in physical units.

4. Let's evaluate mathematical results from the point of view of artificial intelligence.

Let us evaluate the applied methods of multidimensional mathematics - {axiomatics of Mashunin Yu.K., principles of optimality and methods for solving vector problems of mathematical programming}, and compare them with the methods of artificial intelligence.

Using the theory of vector optimization, for an engineering system (in particular, a technological system, a technical system), we obtained.:

1. The optimal solution with equivalent criteria, which includes:

Optimum points are the parameters of the engineering system - $X^o = (x_j^o, j = \overline{1, N})$;

Characteristics (criteria) with equivalent criteria – $F(X^o) = \{f_k(X^o), k = \overline{1, K}\}$;

Relative evaluations with Equivalent Criteria – $\lambda(X^o) = \{\lambda_k(X^o), k = \overline{1, K}\}$, which lie within $\{0 \leq \lambda_k(X^o) \leq 1 \text{ (или 100\%)}, k = \overline{1, K}\}$, and is easily converted into natural units.

2. Optimal solution for a given numerical value (priority) of any criterion (characteristic).

The numerical value of the criterion is formed by the intellect of the system designer.

The solution for a given value of the priority criterion (characteristic) includes:

Optimum points – parameters of an engineering system – with a given priority of any criterion (characteristic): $X^q: f_k(X^q) = \{f_1(X^q), f_2(X^q), f_3(X^q), f_4(X^q)\}, \forall k = \overline{1, K}$.

Formation of indicators (characteristics) at a given priority of any criterion at the optimum point $X^q: \lambda^q = \{\lambda_k^q, k = \overline{1, K}, \lambda_k(X^q) = \frac{f_k(X^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}$.

Formation of relative estimates for a given priority of any criterion at the optimum point $X^q: \lambda^q = \{\lambda_k^q, k = \overline{1, K}, \lambda_k(X^q) = \frac{f_k(X^q) - f_k^0}{f_k^* - f_k^0}, k = \overline{1, K}$:

$\lambda_k(X^q) = \{\lambda_1(X^q), \lambda_2(X^q), \lambda_3(X^q), \lambda_4(X^q), \forall k = \overline{1, K}\}$.

Can these results be obtained by artificial intelligence, which, as a rule, functions on the principle of brute force? The answer is "No." Artificial intelligence can only get an approximate result, and this result must be evaluated by a human. Thus, the developed theory of multidimensional mathematics can be the mathematical apparatus of the computational process of artificial intelligence.

5. Conclusions.

The problem of developing mathematical methods of multidimensional mathematics and making an optimal decision based on them in a complex engineering system based on a certain set of functional characteristics and experimental data is one of the most important tasks of system analysis and design of engineering systems.

The paper presents the theory, methods of multidimensional mathematics in the system of optimal decision-making in problems arising during the study, design and selection of parameters in multiparameter and multifunctional engineering systems, which include technical systems, technological processes, materials. On the basis of multidimensional mathematics, the paper develops and presents a methodology for design automation by: building a mathematical model of an engineering system under conditions of certainty and uncertainty.

The construction of a mathematical and numerical model for the selection of optimal parameters of a complex engineering system is presented on the example of a technological and technical system with four parameters and four characteristics. Modeling and selection of optimal parameters of engineering systems are implemented in two stages. The first stage of work is aimed at solving the choice of optimal parameters of the engineering system in conditions of certainty and



uncertainty with equivalent criteria. The second stage of work is aimed at solving the choice of optimal parameters of the engineering system in conditions of uncertainty, while using the intellectual knowledge of the system designer. The problem is solved and the geometric interpretation of the results of the solution of the N -dimensional engineering system of parameters into a two-dimensional system of parameters is shown.

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